Homework 9

Mathematical Modeling

Due: December 3, 2018

1 Problems for Everybody

1. Consider the Lorenz equations:

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = rx - y - xz \\ \dot{z} = xy - bz \end{cases}$$

where $\sigma > 0$, $r > 0$, and $b > 0$ are constants.

   (a) Show that the fixed points for this system are $(0,0,0)$ and $C_{\pm} = \left(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1\right)$. For what values of r will $C_{\pm}$ exist. **Hint:** To show something is true you can simply substitute into the equations.

   (b) Analyze the local stability of the fixed point $(0,0,0)$. What type of bifurcation occurs in r?

   (c) Show that for $r < 1$ the origin $(0,0,0)$ is globally stable by considering the following Lyapunov function: $V(x,y,z) = \sigma^{-1}x^2 + y^2 + z^2$.

   (d) Show that the characteristic equations for the eigenvalues of the Jacobian matrix at $C_{\pm}$ is

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + (r + \sigma)b\lambda + 2b\sigma(r - 1) = 0.$$

   By seeking solutions of the form $\lambda = i\omega$ show that there is a pair of pure imaginary eigenvalues when

$$r_H = \sigma\left(\frac{\sigma + b + 3}{\sigma - b - b}\right).$$

   Explain why we need to assume $\sigma > b + 1$.

   (e) Show that there is a certain ellipsoidal region E of the form $rx^2 + \sigma y^2 + \sigma(z - 2r)^2 \leq C$ such that all trajectories of the Lorenz equations eventually enter E and stay in there forever.

   (f) Show that the z-axis is an invariant line for the Lorenz equations.

2. In the course we have been using the concept of an attractor without properly defining it. An **attractor** is a closed set A satisfying the following properties:

   - *A is an invariant set*: any trajectory $\mathbf{x}(t)$ that starts in A stays in A for all time.
   - *A attracts an open set of initial conditions:* there is an open set U containing A such that if $\mathbf{x}(0) \in U$, then the distance from $\mathbf{x}(t)$ to A tends to zero as $t \to \infty$. This means that A attracts all trajectories that start sufficiently close to it. That largest such U is called the *basin of attraction*.
   - *A is minimal*: there is no proper subset of A that satisfies the above two conditions.

   Now consider the following system in polar coordinates:

$$\begin{cases} \dot{r} = r(1 - r^2)\dot{\theta} = 1 \end{cases}.$$

   Let D be the disk $x^2 + y^2 \leq 1$.

#1.

Consider the Lorenz equations:

$$\begin{cases} \dot{x} = \sigma(y-x) \\ \dot{y} = rx - y - xz \\ \dot{z} = xy - bz \end{cases}$$

where $\sigma>0$, $r>0$ and $b>0$ are constants.

b.) Analyze the local stability of $(0,0,0)$. What type of bifurcation occurs in r?

Solution:

$$J(0,0,0) = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}.$$

Consequently, the eigenvalues of $J(0,0,0)$ are given by:

$$\lambda_\pm = \frac{-(1+\sigma) \pm \sqrt{(1+\sigma)^2 - 4\sigma(1-r)}}{2}, \quad \lambda_3 = -b$$

Consequently, if $r<1$ then $Re(\lambda_\pm)<0$ and if $r>1$ then $Re(\lambda_+)>0$. Moreover, since $C_\pm = 0$ when $r=1$ it follows that $r=1$ is a transcritical bifurcation.

d.) Show that the characteristic equation for the eigenvalues of the Jacobian matrix at $C_\pm$ is

$$\lambda^3 + (\sigma+b+1)\lambda^2 + (r+\sigma)b\lambda + 2b\sigma(r-1) = 0.$$

Show that there is a pair of imaginary eigenvalues when

$$r_H = \sigma\left(\frac{\sigma+b+3}{\sigma-b-1}\right).$$

Solution:

Calculating it follows that

$$J(x,y) = \begin{bmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{bmatrix}$$

$$\Longrightarrow J(C_\pm) = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \mp\sqrt{b(r-1)} \\ \pm\sqrt{b(r-1)} & \pm\sqrt{b(r-1)} & -b \end{bmatrix}$$

The characterist polynomial is given by:

$$\lambda^3 - I_J \lambda^2 + II_J \lambda - III_J = 0$$

where the principal invariants are given by

$$I_J = Tr(J)$$

$$II_J = \frac{1}{2}\left(Tr(J)^2 - Tr(J^2) \right)$$

$$III_J = det(J).$$

Now,

$$I_J = -\sigma - 1 - b$$

$$III_J = -\sigma(b + (b(r-1)) - \sigma(-b + (b(r-1))$$

$$= -2b\sigma(r-1)$$

$$II_J = (r+\sigma)b$$

$$\Rightarrow \lambda^3 + (\sigma + b + 1)\lambda^2 + (r+\sigma)b\lambda + 2b\sigma(r-1) = 0.$$

If we assume $\lambda = iw$ then

$$-w^3 i - (\sigma+b+1)w^2 + (r+\sigma)b\,iw + 2b\sigma(r-1) = 0$$

By comparing real and imaginary parts it follows that

$$-w^3 + (r+\sigma)bw = 0$$

$$-(\sigma+b+1)w^2 + 2b\sigma(r-1) = 0$$

$$\Rightarrow w = \sqrt{b(r+\sigma)}$$

$$\Rightarrow -(\sigma+b+1)(b(r+\sigma)) + 2b\sigma(r-1) = 0.$$

$$\Rightarrow -(\sigma+b+1)br - b\sigma(\sigma+b+1) + 2b\sigma r - 2b\sigma = 0$$

$$\Rightarrow b(\sigma-b-1)r - b\sigma(\sigma+3\sigma+1) = 0$$

$$\Rightarrow r = \frac{\sigma(\sigma+3\sigma+1)}{\sigma-b-1}.$$

e.) Show that there is a certain ellipsoidal region of the form $rx^2+\sigma y^2+\sigma(z-2r)^2 \le C^2$ such that all trajectories of the Lorenz equations eventually enter E and stay in there forever.

Solution:

Let $\rho = (rx^2+\sigma y^2+\sigma(z-2r)^2)^{1/2}$ denonote the elliptical distance from $(0,0,2r)$. Calculating it follows that:

$$\frac{d}{dt}\rho^2 = 2\rho\dot{\rho} = 2rx\dot{x}+2\sigma y\dot{y}+2\sigma(z-2r)\dot{z}$$
$$= 2rx\sigma(y-x)+2\sigma y(rx-y-xz)+2\sigma(z-2r)(xy-bz)$$
$$= -2r\sigma x^2+4r\sigma xy-2\sigma y^2-2\sigma xyz+2\sigma xyz-2\sigma bz^2 -4\sigma rxy-4\sigma rbz$$
$$= -2r\sigma x^2-2\sigma y^2-2\sigma bz^2-4\sigma rbz$$
$$= -2r\sigma x^2-2\sigma y^2-2\sigma b(z+2r)^2+4r^2.$$

Therefore, for sufficiently large x, y, z, $\frac{d\rho}{dt}<0$ on the boundary of the ellipsoid $rx^2+\sigma y^2+\sigma(z-2r)^2=C^2$.

f.) Show that the z-axis is invariant for the Lorenz equations.

Solution:

If $x, y = 0$ then

$$\dot{x}=0$$
$$\dot{y}=0$$

and consequently the z-axis is invariant.

#2.

Consider the following system in polar coordinates:

$$\begin{cases} \dot{r} = r(1-r^2), \\ \dot{\theta} = 1 \end{cases}$$

Let D be the disk $x^2+y^2 \leq 1$.

a.) Is D an invariant set?

Solution:

Yes.

b.) Does D attract an open set of initial conditions?

Solution:

Yes

c.) Is D an attractor?

Solution:

No. It is not minimal.

d) Repeat part c for the circle $x^2+y^2=1$.

Solution:

Yes, with basin of attraction $\mathbb{R}^2$.