Bonus Homework

Mathematical Modeling

Due: September 26, 2018

1 Problems for Everybody

For each of the following problems sketch all qualitatively different phase portraits that occur as r is varied. Sketch a bifurcation diagram of fixed points x^* versus r. **If possible determine the bifurcation point explicitly**. In each bifurcation diagram determine what type of bifurcation occurs.

1. $\dot{x} = 5 - re^{-x^2}$.
2. $\dot{x} = rx - 4x^2$.
3. $\dot{x} = rx + x^3 - x^5$.
4. $\dot{x} = r + x - \ln(1 + x)$.

Bonus Homework:

#1.

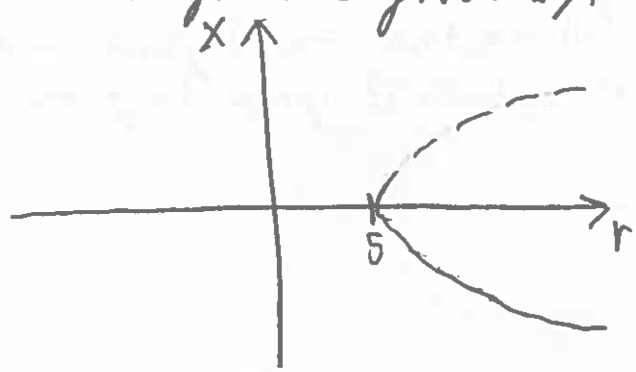
$$\dot{x} = 5 - re^{-x^2}$$

Solution:

The function $f(x) = re^{-x^2}$ has a maximum value of r at $x=0$. Consequently, $g(x) = 5 - re^{-x^2}$ satisfies the following properties:

1. $g(0)$ is a local minimum with minimum value $g(0) = 5 - r$.
2. $\lim_{x \to \pm\infty} g(x) = 5$.
3. g is an even function
4. g is monotone increasing for $x \geq 0$.

Therefore, for $r>5$ g has two roots and thus the resulting bifurcation diagram is given by:

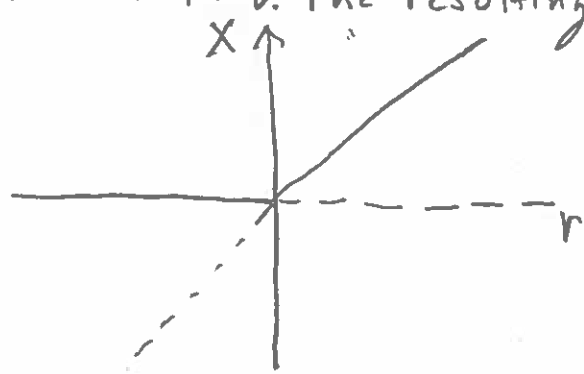


#2.

$$\dot{x} = rx - 4x^2$$

Solution:

The fixed points are given by $x^* = 0$, $x^* = r/4$. If we let $f(x) = rx - 4x^2$ then $f'(0) = r$. Therefore, there is a transcritical bifurcation at $r=0$. The resulting bifurcation is given by:



#3.

$$\dot{x} = rx + x^3 - x^5.$$

Solution:

The fixed points for this system are given by:

$$x(r + x^2 - x^4) = 0$$

$$\Rightarrow x^* = 0, \quad x^4 - x^2 - r = 0$$

$$\Rightarrow x^* = 0, \quad x^2 = \frac{1 \pm \sqrt{1+4r}}{2}$$

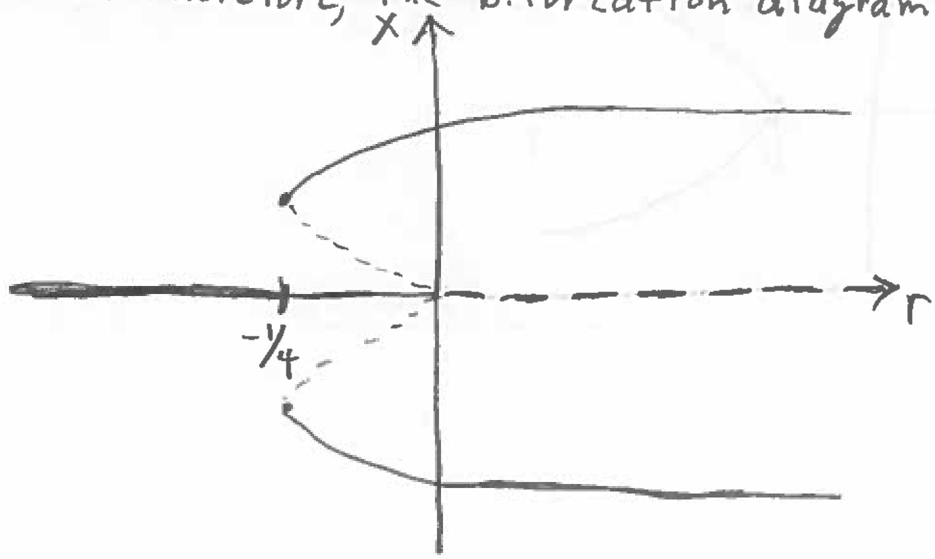
$$\Rightarrow x^* = 0, \quad x^* = \pm \frac{1}{\sqrt{2}} \sqrt{1 \pm \sqrt{1+4r}}$$

Bifurcations will occur when

$$1 + 4r = 0 \quad \text{or} \quad 1 - \sqrt{1+4r} = 0$$

$$\Rightarrow r = -1/4 \quad \text{or} \quad r = 0$$

Since, $\lim_{x\to\infty} rx + x^3 - x^5 = -\infty$ it follows that the right-most fixed point is stable. Therefore, the bifurcation diagram is given by:



#4.

$\dot{x} = r + x - \ln(1+x)$.

Solution:

Let $f(x) = r + x - \ln(1+x)$. Since r vertically shifts the graph of $x - \ln(1+x)$ it follows that a saddle node bifurcation will occur when $f'(x^*) = 0$ and $f(x^*) = 0$. Calculating:

$$f'(x^*) = 1 - \frac{1}{1+x}$$

$$\Rightarrow f'(x^*) = 0 \Rightarrow x^* = 0$$

Furthermore,

$$f(0) = r$$

and thus the bifurcation occurs when $r = 0$. The resulting bifurcation diagram is thus given by: