MST 205
Fall 2021
Exam #2
10/14/21

Name (Print): Key

The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Short answer questions:** Questions labeled as **"Short Answer"** can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit.** An incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Problem	Points	Score
1	20	
2	15	
3	15	
4	15	
5	20	
6	15	
Total:	100	

Do not write in the table to the right.

1. (20 points) (**Short Answer**) Determine if the following statement is correct (**C**) or incorrect (**I**). Just circle **C** or **I**. No need to show any work. In order for a statement to be correct it must be true in all cases. In these problems, $F(-\infty, \infty)$ denotes the vector space of all functions defined on $(-\infty, \infty)$ with the standard operations of addition and multiplication. **Hint:** If the answer to these questions does not come quickly just move on and come back later.

C **I** The set of all pairs of real numbers (x, y) with the operations

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \text{ and } k(x, y) = (kx, y)$$

is a vector space.

C **I** The set of all pairs of real numbers of the form $(x, 0)$ with the standard operations of addition and multiplication is a subspace of $\mathbb{R}^2$.

C **I** The set of all pairs of real numbers of the form (x, x) with the standard operations of addition and multiplication is a subspace of $\mathbb{R}^2$.

C **I** The set of all pairs of real numbers of the form (x, y), where $x \geq 0$, with the standard operations of addition and multiplication is a subspace of $\mathbb{R}^2$.

C **I** The set of functions such that $f(x) \leq 0$ for all x is a subspace of $F(-\infty, \infty)$.

C **I** The set of functions such that $f(0) = 0$ is a subspace of $F(-\infty, \infty)$.

C **I** The set of all constant functions is a subspace of $F(-\infty, \infty)$.

C **I** If A is $n \times n$ matrix satisfying $\det(A) \neq 0$ then $NS(A) = \{0\}$.

C **I** If A is $n \times n$ matrix satisfying $\det(A) \neq 0$ then $CS(A) = \mathbb{R}^n$.

C **I** If A is an $m \times n$ matrix with $m < n$, then the columns of A are linearly dependent.

2. (15 points) Determine whether the vector

$$\mathbf{v} = \begin{bmatrix} 1 \\ -5 \\ -3 \end{bmatrix}$$

is in the span of the vectors

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

You must show all of your work to receive full credit for this problem.

If $\vec{v} \in \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ then there exists c_1, c_2, c_3

such that

$$c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ -3 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ -1 & 1 & 0 & -5 \\ 0 & 1 & 1 & -3 \end{array}\right] \begin{array}{l} \\ +R1 \\ \\ \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 2 & 2 & -4 \\ 0 & 1 & 1 & -3 \end{array}\right] \begin{array}{l} \\ \\ -2R3 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 2 & 2 & -4 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

$5 \neq 0$ and thus $\vec{v} \notin \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

3. (15 points) Determine if the polynomials x^2+x+1, x^2-x+1, and x^2-1 form a basis for P_2, the vector space of all quadratic polynomials. You must show all of your work to receive full credit for this problem.

These functions will form a basis if they are linearly independent.

$$\Rightarrow c_1(x^2+x+1)+c_2(x^2-x+1)+c_3(x^2-1)=0$$

$$\Rightarrow (c_1+c_2+c_3)=0$$
$$(c_1-c_2)=0$$
$$c_1+c_2-c_3=0$$

$$\Rightarrow c_1=c_2$$

$$\Rightarrow 2c_1+c_3=0$$
$$2c_1-c_3=0$$

$$\Rightarrow c_3=2c_1$$
$$c_3=-2c_1$$

$$\Rightarrow c_1=c_2=c_3=0.$$

These functions do form a basis.

4. (15 points) Let α be the basis given by

$$\left\{\begin{bmatrix}2\\-4\end{bmatrix}, \begin{bmatrix}3\\8\end{bmatrix}\right\}.$$

(a) (10 points) Find $[v]_\alpha$ if

$$v = \begin{bmatrix}1\\1\end{bmatrix}$$

$$\begin{bmatrix}1\\1\end{bmatrix} = c_1\begin{bmatrix}2\\-4\end{bmatrix} + c_2\begin{bmatrix}3\\8\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}2 & 3 & 1\\-4 & 8 & 1\end{bmatrix} {\scriptstyle +2R1} \Rightarrow \begin{bmatrix}2 & 3 & 1\\0 & 14 & 3\end{bmatrix}$$

$$\Rightarrow c_2 = 3/14$$

$$2c_1 + 9/14 = 1 \quad \Rightarrow [V]_\alpha = \begin{bmatrix}5/28\\3/14\end{bmatrix}$$

$$2c_1 = 5/14$$

(b) (5 points) Find w if

$$[w]_\alpha = \begin{bmatrix}1\\1\end{bmatrix}.$$

$$W = 1\cdot\begin{bmatrix}2\\-4\end{bmatrix} + 1\cdot\begin{bmatrix}3\\8\end{bmatrix} = \begin{bmatrix}5\\4\end{bmatrix}.$$

5. (20 points) If

$$A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 1 & 1 & 1 & -2 \\ 3 & 3 & 2 & -1 \end{bmatrix},$$

find a basis for $NS(A)$, $RS(A)$, $CS(A)$ and determine the rank of A.

$$\left[\begin{array}{cccc|c|c} 1 & 1 & 0 & 3 & 0 & a \\ 1 & 1 & 1 & -2 & 0 & b \\ 3 & 3 & 2 & -1 & 0 & c \end{array}\right] \begin{array}{l} \\ -R1 \\ -3R1 \end{array}$$

$$\left[\begin{array}{cccc|c|c} 1 & 1 & 0 & 3 & 0 & a \\ 0 & 0 & 1 & -5 & 0 & b-a \\ 0 & 0 & 2 & -10 & 0 & c-3a \end{array}\right] \begin{array}{l} \\ \\ -2R2 \end{array}$$

$$\left[\begin{array}{cccc|c|c} 1 & 1 & 0 & 3 & 0 & a \\ 0 & 0 & 1 & -5 & 0 & b-a \\ 0 & 0 & 0 & 0 & 0 & c-3a-2b+2a \end{array}\right]$$

$$RS(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -5 \end{bmatrix}\right\},$$

$$\text{Rank}(A) = 3.$$

If $\vec{x} \in NS(A)$ then

$$x_3 = 5x_4$$

$$x_1 + x_2 + 3x_4 = 0$$

$$\Rightarrow x_1 = -x_2 - 3x_4$$

$$\Rightarrow \vec{x} = \begin{bmatrix} -x_2 - 3x_4 \\ x_2 \\ 5x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 5 \\ 1 \end{bmatrix}$$

$$NS(A) = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 5 \\ 1 \end{bmatrix}\right\}.$$

If $\vec{y} \in CS(A)$ then

$$c = a + 2b$$

$$\Rightarrow \vec{y} = \begin{bmatrix} a \\ b \\ a+2b \end{bmatrix} = a \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \Rightarrow CS(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\right\}.$$

6. (15 points) Suppose A is a 3×5 matrix.

   (a) (5 points) (**Short Answer:**) What is the largest possible value for the dimension of the rowspace of A?

   Three

   $$\begin{bmatrix} 1 & & & * \\ 0 & 1 & & \\ 0 & 0 & 1 & \end{bmatrix}$$

   (b) (5 points) (**Short Answer:**) What is the smallest possible value for the dimension of the nullspace of A?

   Two

   (c) (5 points) (**Short Answer:**) What is the largest possible value for the dimension of the columnspace of A?

   Three.