pg. 93, #6

$$c_1\begin{bmatrix}0\\4\\-1\end{bmatrix}+c_2\begin{bmatrix}1\\5\\-3\end{bmatrix}+c_3\begin{bmatrix}1\\-3\\-1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow\begin{array}{c}\left[\begin{array}{ccc|c}0&1&1&0\\4&5&-3&0\\-1&-3&-1&0\end{array}\right]\end{array}\begin{array}{l}\\+4R3\\\times -1\end{array}\Rightarrow\left[\begin{array}{ccc|c}0&1&1&0\\0&-7&-7&0\\1&3&1&0\end{array}\right]\begin{array}{l}\\+7R1\\\\\end{array}$$

$$\left[\begin{array}{ccc|c}0&1&1&0\\0&0&0&0\\1&3&1&0\end{array}\right]$$

These vectors are linearly ~~in~~dependent.

pg. 93, #8

$$c_1\begin{bmatrix}1&-1\\2&1\end{bmatrix}+c_2\begin{bmatrix}1&0\\0&-1\end{bmatrix}+c_3\begin{bmatrix}1&-2\\0&1\end{bmatrix}+c_4\begin{bmatrix}3&-3\\2&1\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$\Rightarrow\begin{aligned}c_1+c_2+c_3+3c_4&=0\\-c_1+0c_2-2c_3-3c_4&=0\\2c_1+0c_2+0c_3+2c_4&=0\\c_1-c_2+c_3+c_4&=0\end{aligned}$$

$$\Rightarrow\left[\begin{array}{cccc|c}1&1&1&3&0\\-1&0&-2&-3&0\\2&0&0&2&0\\1&-1&1&1&0\end{array}\right]\begin{array}{l}\\+R1\\-2R1\\-R1\end{array}\Rightarrow\left[\begin{array}{cccc|c}1&1&1&3&0\\0&1&-1&0&0\\0&-2&-2&-4&0\\0&-2&0&-2&0\end{array}\right]\begin{array}{l}\\\\+2R2\\+2R2\end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -4 & -4 & 0 \\ 0 & 0 & -2 & -2 & 0 \end{array}\right]$$

Row four is a multiple of row 3, so it is not linearly independent, i.e., linearly dependent.

#18.

Show that $\{x^3+x, x^2-x, x+1, x^3+1\}$ form a basis for P_3

Suppose

$$c_1(x^3+x)+c_2(x^2-x)+c_3(x+1)+c_4(x^3+1)=0$$

$$\Rightarrow (c_1+c_4)x^3+c_2x^2+(c_1-c_2+c_3)x+c_3+c_4=0$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \begin{array}{c} \\ \\ -R1 \\ \\ \end{array} \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \begin{array}{c} \\ \\ +R2 \\ \\ \end{array}$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

These polynomials are linearly independent. A similiar calculation shows it spans P_3. Therefore, these vectors form a basis.

#24

If β is the basis $\left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix} \right\}$

If $\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix}$

then

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & -1 \\ -1 & 3 & -4 & 1 \\ 0 & -1 & -1 & 0 \end{array}\right] +2R2 \Rightarrow \left[\begin{array}{ccc|c} 0 & 7 & -7 & 1 \\ -1 & 3 & -4 & 1 \\ 0 & -1 & -1 & 0 \end{array}\right] +7R3$$

$$\Rightarrow \left[\begin{array}{ccc|c} 0 & 0 & -14 & 1 \\ -1 & 3 & -4 & 1 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

$\Rightarrow c_3 = -1/14$

$-c_2 - c_3 = 0$

$\Rightarrow c_2 = 1/14$

$-c_1 + \frac{3}{14} + \frac{4}{14} = 1$

$-c_1 = 1 - 1/2$

$c_1 = -1/2$

Therefore

$[\vec{v}]_\beta = \begin{bmatrix} -1/2 \\ +1/14 \\ -1/14 \end{bmatrix}$

Now, if $[\vec{v}]_\beta = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ then

$\vec{v} = -\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}$

#28

Suppose $\vec{v}_{i_1}, \vec{v}_{i_2}, \ldots, \vec{v}_{i_m}$ is a subset of $\vec{v}_1, \ldots, \vec{v}_n$ and consider the equation

$$c_{i_1}\vec{v}_{i_1} + \ldots + c_{i_m}\vec{v}_{i_m} = 0.$$

If this equation has a nontrivial solution then so does

$$c_1\vec{v}_1 + \ldots + c_n\vec{v}_n$$

which is a contradiction