Homework #7

#2.

a.) $\begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & -1 \\ 4 & 0 & 8 \end{bmatrix} \begin{matrix} \\ -R1 \\ -4R1 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & -1 \\ 0 & -8 & 8 \end{bmatrix}$

These vectors are a basis since they are linearly independent.

b.) $\begin{bmatrix} 3 & -1 & 3 \\ 2 & -1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{matrix} -3R3 \\ -2R3 \\ \\ \end{matrix} \Rightarrow \begin{bmatrix} 0 & -1 & -3 \\ 0 & -1 & -3 \\ 1 & 0 & 2 \end{bmatrix}$

These vectors are linearly dependent and thus do not form a basis.

c.) Not a basis

d.) Not a basis.

#10

a.) Row reducing it follows that:

$\left[\begin{array}{cccc|c} 2 & -1 & 3 & 4 & 0 \\ 1 & 0 & -1 & 3 & 0 \end{array}\right] \begin{matrix} -2R1 \\ \\ \end{matrix} \Rightarrow \left[\begin{array}{cccc|c} 0 & -1 & 5 & -2 & 0 \\ 1 & 0 & -1 & 3 & 0 \end{array}\right]$

$\Rightarrow x_2 = 5x_3 - 2x_4$

$x_1 = x_3 - 3x_4$

$\Rightarrow$ If $\vec{x} \in NS(A)$ then

$\vec{x} = x_3 \begin{bmatrix} 1 \\ 5 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

Therefore, a basis for the nullspace is

$$\left\{ \begin{bmatrix} 1 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

b.) A basis for the rowspace is clearly:

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 3 \end{bmatrix} \right\}$$

c) A basis for the columnspace is clearly:

$$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}.$$

d) $\mathrm{Rank}(A) = 2$.

#12.

Here I am going to consider solving all problems at once. Suppose $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_5 \end{bmatrix} \in NS(A)$ and $\vec{a} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in CS(A)$ this yields the doubly augmented matrix:

$$\left[\begin{array}{ccccc|c|c} 1 & -1 & -1 & 2 & 0 & 0 & a \\ -2 & 1 & 1 & -1 & 0 & 0 & b \\ 1 & 1 & -2 & 1 & 1 & 0 & c \end{array}\right] \begin{array}{l} \\ +2R1 \\ -R1 \end{array} \Rightarrow \left[\begin{array}{ccccc|c|c} 1 & -1 & -1 & 2 & 0 & 0 & a \\ 0 & -1 & -1 & 3 & 0 & 0 & b+2a \\ 0 & 2 & -1 & -1 & 1 & 0 & c-a \end{array}\right] \begin{array}{l} \\ \\ +2R2 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccccc|c|c} 1 & -1 & -1 & 2 & 0 & 0 & a \\ 0 & -1 & -1 & 3 & 0 & 0 & b+2a \\ 0 & 0 & -5 & 5 & 1 & 0 & c+2b+3a \end{array}\right]$$

Therefore,

$-5x_3 + 5x_4 + x_5 = 0$

$\Rightarrow x_5 = 5x_3 - 5x_4$

$-x_2 - x_3 + 3x_4 = 0$

$\Rightarrow x_2 = -x_3 + 3x_4$

$x_1 - x_2 - x_3 + 2x_4 = 0$

$\Rightarrow x_1 + x_3 - 3x_4 - x_3 + 2x_4 = 0$

$\Rightarrow x_1 = x_4$

Therefore,

$$\vec{x} = \begin{bmatrix} x_4 \\ -x_3 + 3x_4 \\ x_3 \\ x_4 \\ 5x_3 - 5x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 5 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \\ -5 \end{bmatrix}$$

Furthermore, there is no restriction on $\vec{a}$.

(a) $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \\ -5 \end{bmatrix} \right\}$

b.) $\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \\ 1 \end{bmatrix} \right\}$

c.) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

d) Rank(A) = 3.

#14.

If $\vec{a}=\begin{bmatrix}a\\b\\c\end{bmatrix} \in \operatorname{span}\left\{\begin{bmatrix}0\\-2\\1\end{bmatrix},\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}2\\0\\-1\end{bmatrix},\begin{bmatrix}4\\-2\\-1\end{bmatrix}\right\}$ then there exists $c_1, c_2, c_3, c_4 \in \mathbb{R}$ such that

$$c_1\begin{bmatrix}0\\-2\\1\end{bmatrix}+c_2\begin{bmatrix}1\\-1\\0\end{bmatrix}+c_3\begin{bmatrix}2\\0\\-1\end{bmatrix}+c_4\begin{bmatrix}4\\-2\\-1\end{bmatrix}=\begin{bmatrix}a\\b\\c\end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cccc|c}0&1&2&4&a\\-2&-1&0&-2&b\\1&0&-1&-1&c\end{array}\right]+2R3 \Rightarrow \left[\begin{array}{cccc|c}0&1&2&4&a\\0&-1&-2&-4&b+2c\\1&0&-1&-1&c\end{array}\right]+R2$$

$$\Rightarrow \left[\begin{array}{cccc|c}0&0&0&0&a+b+2c\\0&-1&0&0&b+2c\\1&0&-1&-1&c\end{array}\right]$$

To ensure a solution $a+b+2c=0 \Rightarrow a=-b-2c$. Therefore anything in the span can be expressed as

$$\vec{a}=\begin{bmatrix}-b-2c\\b\\c\end{bmatrix}=b\begin{bmatrix}-1\\1\\0\end{bmatrix}+c\begin{bmatrix}-2\\0\\1\end{bmatrix}.$$

Therefore, a basis is given by $\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix},\begin{bmatrix}-2\\0\\1\end{bmatrix}\right\}$.

#18.

We know the rank of $A=\dim(RS(A))=\dim(CS(A))$. The largest the rank can be is n. Therefore, the largest the $\dim(RS(A))$ can be is $n<m$. Consequently one of the rows must be linearly dependent.