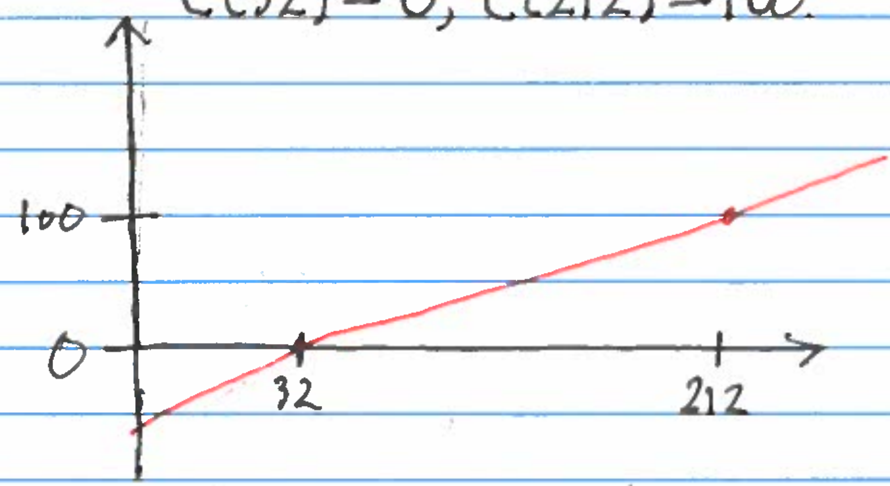
Section 1.1: Systems of Linear Equations

Example:

Convert $50°F$ to degrees celsius. Define
$C(F) \leftarrow$ function takes in $°F$ and outputs $°C$.

What we know:
$C(32)=0$, $C(212)=100$.



$C(F)=mF+b$, What is $C(50)$??

$* \ C(32)=m32+b=0$

$* \ C(212)=m212+b=100$

We need to solve:

$$\left.\begin{aligned} 32m+b&=0 \\ 212m+b&=100 \end{aligned}\right\}$$ System of two linear equations.

Method 1:

$32m+b=0$

$\Rightarrow b=-32m$

$\Rightarrow 212m-32m=100$

$\Rightarrow 180m=100$

$\Rightarrow m=\frac{100}{180}=5/9 \Rightarrow C(F)=\frac{5}{9}(F-32)$

$b=-32\cdot\frac{5}{9}$

$C(50)=\frac{5}{9}\cdot 18=10$

Method 2:

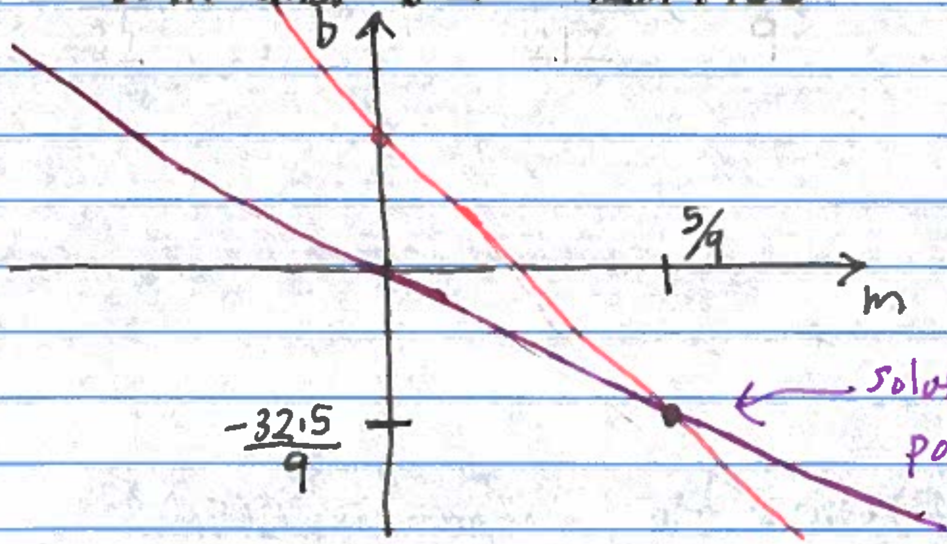
$$32m + b = 0$$
$$212m + b = 100$$

Subtracting first from second $\Rightarrow 180m = 100 \rightarrow m = 5/9$

Therefore, $32 \cdot 5/9 + b = 0 \Rightarrow b = -\frac{32 \cdot 5}{9}$.

Method 3:

$b = -32m$ and $b = -212m + 100$



Solution lies at point of intersection.

Example:

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

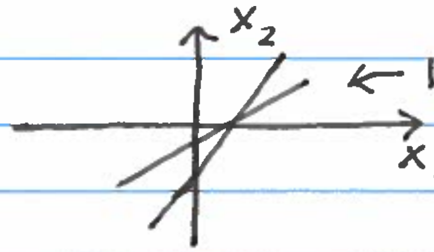
System of two linear equations.

$$\Rightarrow x_2 = \underbrace{-\frac{a_{11}}{a_{12}}}_{\text{slope } m_1} x_1 + \frac{b_1}{a_{12}} \quad x_2\text{-intercept}$$

$$x_2 = \underbrace{-\frac{a_{21}}{a_{22}}}_{\text{slope } m_2} x_1 + \frac{b_2}{a_{22}} \quad x_2\text{-intercept}$$

Case 1:

$\frac{-a_{11}}{a_{12}} \neq \frac{-a_{21}}{a_{22}}$

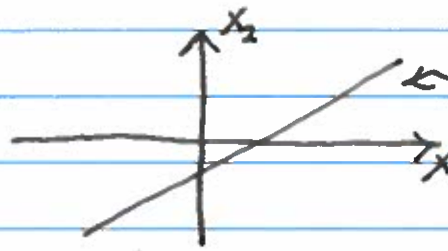


← lines intersect 1-solution.

Case 2:

$\frac{-a_{11}}{a_{12}} = \frac{-a_{21}}{a_{22}}$

$b_1/a_{12} = b_2/a_{22}$

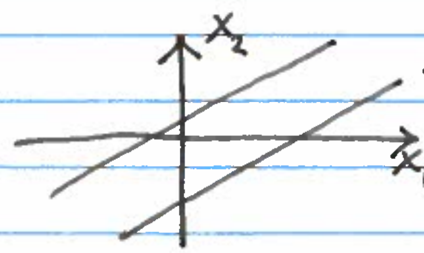


← lines on top of each other. infinite number of solutions.

Case 3:

$\frac{-a_{11}}{a_{12}} = \frac{-a_{21}}{a_{22}}$

$b_1/a_{12} = b_2/a_{22}$



← lines are parallel. No solutions.

Example:

Solve the following system of equations!

(R1)	$x+y-z=0$		$x+y-z=0$
(R2)	$2x+3y-2z=6$	$-2R1 \rightarrow$	$(3y-2y)+(-2z+2z)=6$
(R3)	$x+2y+2z=10$	$-R1$	$(2y-y)+(2z+z)=10$

$\Rightarrow$ $x+y-z=0$
$y=6$
$y+3z=10$ $-R2$

$\Rightarrow$ $x+y-z=0$
$y=6$
$3z=4$

$\Rightarrow$ $z=4/3$

$y=6$

$x+6-4/3=0$

$3x+18-4=0$

$x=-14/3$

Example:

$$\begin{aligned} x_1 + x_2 - x_3 + 2x_4 &= 1 \\ x_1 + x_2 - x_3 - x_4 &= -1 \\ x_1 + 2x_2 + x_3 + 2x_4 &= -1 \\ 2x_1 + 2x_2 + x_3 + x_4 &= 2 \end{aligned}$$

← The variables are irrelevant.

$$\begin{array}{r} \\ -R1 \\ -R1 \\ -2R1 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 1 \\ 1 & 1 & -1 & -1 & -1 \\ 1 & 2 & 1 & 2 & -1 \\ 2 & 2 & 1 & 1 & 2 \end{array}\right]$$

Augmented Matrix

Coefficient Matrix.

Goal: elimate coefficients to isolate variables.

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & -3 & -2 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 3 & -3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 1 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & -3 & -2 \\ 0 & 0 & 3 & -3 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 1 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & -3 & 0 \end{array}\right] \begin{array}{l} \\ \\ R3/3 \\ R4/-3 \end{array} \longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 1 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow \begin{aligned} x_1 + x_2 - x_3 + 2x_4 &= 1 \\ x_2 + 2x_3 \quad &= -2 \\ x_3 - x_4 &= 0 \\ x_4 &= 0 \end{aligned}$$

$$\Rightarrow x_4 = 0,\ x_3 = 0,\ x_2 = -2,\ x_1 = 3.$$

Example:

$$\begin{aligned} 2x+3y-4z&=3 \\ 2x+3y-2z&=3 \\ 4x+6y-8z&=6 \end{aligned} \rightarrow \begin{array}{c} \left[\begin{array}{cccc} 2 & 3 & -4 & 3 \\ 2 & 3 & -2 & 3 \\ 4 & 6 & -8 & 6 \end{array}\right] \end{array} \begin{array}{l} \\ -R1 \\ -2R1 \end{array}$$

$$\rightarrow \left[\begin{array}{cccc} 2 & 3 & -4 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$2x+3y-4z=3$$

$$2z=0$$

$0=0 \rightarrow$ No useful information.

$\Rightarrow z=0$,

$$2x+3y=3$$

$$\Rightarrow y=-\frac{2}{3}x+1$$

Example:

Determine Conditions on a, b, c so that the following system of equations has a solution:

$$\begin{aligned} 2x-y+3z&=a \\ x-3y+2z&=b \\ x+2y+z&=c \end{aligned} \Rightarrow \left[\begin{array}{cccc} 2 & -1 & 3 & a \\ 1 & -3 & 2 & b \\ 1 & 2 & 1 & c \end{array}\right] \begin{array}{l} -2R2 \\ \\ -R2 \end{array} \Rightarrow \left[\begin{array}{cccc} 0 & 5 & -1 & a-2b \\ 1 & -3 & 2 & b \\ 0 & 5 & -1 & c-b \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc} 0 & 5 & -1 & a-2b \\ 1 & -3 & 2 & b \\ 0 & 0 & 0 & c+b-a \end{array}\right]$$

For these equations to be consistent we need $c+b-a=0$.

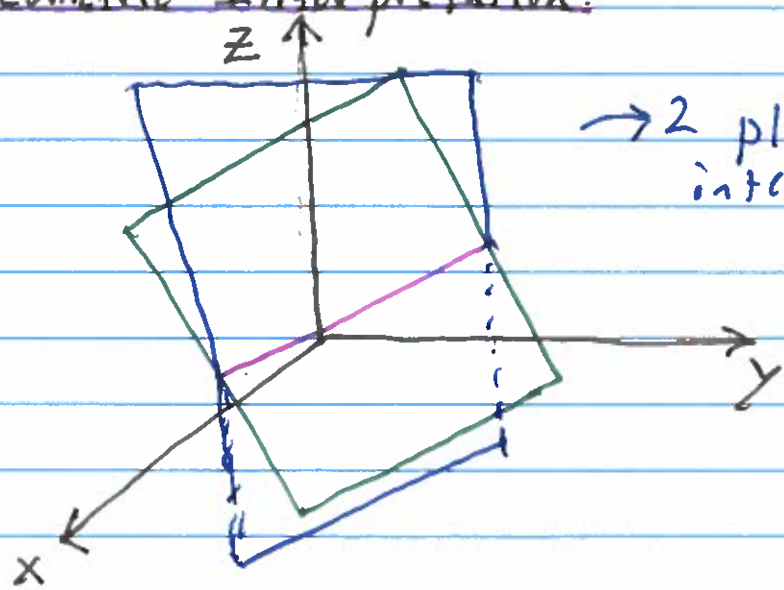
Example:

Does the following system have nontrivial solutions?

$$\begin{aligned} x-y+z&=0 \\ 2x+y+2z&=0 \\ 3x-5y+3z&=0 \end{aligned} \Rightarrow \begin{array}{c} \left[\begin{array}{cccc} 1 & -1 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 3 & -5 & 3 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ -2R1 \\ -3R1 \end{array}$$

$$\Rightarrow \left[\begin{array}{cccc} 1 & -1 & 1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} y=0 \\ z=\text{anything} \\ x=-z. \end{array}$$

Geometric Interpretation:



→ 2 planes intersect in a line.