Section 2.4: Dimension, nullspace, row space, column space

Lemma:

If $\vec{v}_1, \dots, \vec{v}_n$ are a basis and $\vec{w}_1, \dots, \vec{w}_m \in V$ with $m > n$ then $w_1, \dots, w_m$ are linearly dependent. If $m < n$ then $\vec{w}_1, \dots, \vec{w}_m$ do not span V.

Definition - If a vector space V has a basis of n vectors, we say the dimension of V is n.

Example:

1. $\dim(\mathbb{R}^n) = n \Rightarrow$ basis $= \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$
2. $\dim(\mathbb{R}^{n \times m}) = nm \Rightarrow$ basis $= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.
3. $\dim(P_n) = n+1 \Rightarrow$ basis $= \{1, x, x^2\}$.

Theorem - Suppose V is a vector space of dimension n.

1. If $\{\vec{v}_1, \dots, \vec{v}_n\}$ are linearly independent, then $\{\vec{v}_1, \dots, \vec{v}_n\}$ forms a basis for V.
2. If $\{\vec{v}_1, \dots, \vec{v}_n\}$ span V, then $\{\vec{v}_1, \dots, \vec{v}_n\}$ form a basis for V.

Example:

$\{x^2-1, x^2+1, x+1\}$ is a basis for P_2.

linear ind.:

$$c_1(x^2-1) + c_2(x^2+1) + c_3(x+1) = 0$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{matrix} \\ \\ +R1 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \Rightarrow c_1 = c_2 = c_3 \text{ only solution}$$

__Nullspace__

$NS(A) = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = 0\}.$

__Example:__

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 & 0 \\ 1 & 1 & 0 & 4 & 1 \\ 1 & 4 & -3 & 1 & -2 \end{bmatrix} \begin{matrix} \\ -R1 \\ -R1 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 2 & -1 & 3 & 0 \\ 0 & -1 & 1 & 1 & 1 \\ 0 & 2 & -2 & -2 & -2 \end{bmatrix} \begin{matrix} \\ /-1 \\ /2 \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 & 3 & 0 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 & -1 \end{bmatrix} \begin{matrix} \\ \\ -R2 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 2 & -1 & 3 & 0 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

x_5 = anything

$x_2 - x_3 - x_4 - x_5 = 0$

$\Rightarrow x_2 = x_3 + x_4 + x_5$

$x_1 - 2x_2 - x_3 + 3x_4 = 0$

$x_1 - 2(x_3 + x_4 + x_5) - x_3 + 3x_4 = 0$

$\Rightarrow x_1 = 3x_3 - x_4 + 2x_5$

If $\vec{v} \in NS(A)$ then

$$\vec{v} = \begin{bmatrix} 3x_3 - x_4 + 2x_5 \\ x_3 + x_4 + x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \rightarrow x_3 \begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow NS(A) = \text{span}\left(\begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

$\dim(NS(A)) = 3.$

Coordinates:

$$\vec{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \vec{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\vec{v} = \begin{bmatrix}1\\3\\7\end{bmatrix} = 1\vec{e}_1 + 3\vec{e}_2 + 7\vec{e}_3$$

Express $\vec{v}$ relative to the basis β:

$$\begin{bmatrix}1\\6\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix}$$

$$\begin{bmatrix}1\\3\\7\end{bmatrix} = c_1\begin{bmatrix}1\\0\\1\end{bmatrix} + c_2\begin{bmatrix}1\\1\\1\end{bmatrix} + c_3\begin{bmatrix}-1\\1\\1\end{bmatrix}$$

$$\Rightarrow \begin{aligned} c_1 + c_2 - c_3 &= 1 \\ c_2 + c_3 &= 3 \\ c_1 + c_2 + c_3 &= 7 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 3 \\ 1 & 1 & 1 & 7 \end{array}\right] \begin{array}{l} \\ \\ -R1 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 2 & 6 \end{array}\right]$$

$$\Rightarrow c_3 = 3$$

$$c_2 + 3 = 3$$

$$c_2 = 0$$

$$c_1 + 0 - 3 = 1$$

$$c_1 = 4$$

$$\begin{bmatrix}1\\3\\7\end{bmatrix} = \begin{bmatrix}4\\0\\3\end{bmatrix}_\beta$$

Color basis.

One basis — red, blue, yellow

Second Basis — red, blue, green

Third Basis — cyan, magenta, yellow

orange = red + yellow.
= red + green+blue mix.

Example:

$\vec{v} = x+1$

$\gamma = \{x^2+x-3,\ x-5,\ 3\}$

$x+1 = c_1(x^2+x-3) + c_2(x-5) + 3c_3$

$\Rightarrow 0 = c_1 x^2$

$x = c_1 x + c_2 x$

$1 = -3c_1 - 5c_2 + 3c_3$

$\Rightarrow c_1 = 0$

$c_2 = 1$

$1 = -5 + 3c_3$

$\Rightarrow c_3 = 3$

$$\Rightarrow x+1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}_\gamma$$

Row space:

$RS(A)$ = space spanned by rows of A

$$RS\left(\begin{bmatrix} 1 & 2 & -1 & 3 & 0 \\ 1 & 1 & 0 & 4 & 1 \\ 1 & 4 & -3 & 1 & -2 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}\right\}$$

$\dim(RS(A)) = 2.$

$\text{Rank}(A) = \dim(RS(A))$

$\text{Nullity}(A) = \dim(NS(A))$

Theorem – $\text{Rank}(A) + \text{Nullity}(A) = \#$ of columns of A.

Column Space

$CS(A)$ = space spanned by columns of A

$$CS\left(\begin{bmatrix} 1 & 2 & -1 & 3 & 0 \\ 1 & 1 & 0 & 4 & 1 \\ 1 & 4 & -3 & 1 & 2 \end{bmatrix}\right) = RS\left(\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ -1 & 0 & -3 \\ 3 & 4 & 1 \\ 0 & 1 & -2 \end{bmatrix}\right) \begin{matrix} \\ -2R1 \\ +R1 \\ -3R1 \\ \\ \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix} \begin{matrix} \\ \\ +R2 \\ +R2 \\ +R2 \end{matrix} \quad \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow CS(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}\right\}$$

Range(A)

$Ra(A) = \{\vec{w} ;$ there exists $\vec{x}$ such that $A\vec{x} = \vec{w}\}$

$Ra(A) = CS(A)$

proof:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{c|c|c} c_1 & \cdots & c_n \end{array}\right] = A \Rightarrow A\vec{e}_i = \vec{c}_i$$

If $\vec{w} \in Ra(A) \Rightarrow$ there exists $\vec{v}$ such that

$$A\vec{v} = \vec{w}$$

$$\Rightarrow A(a_1\vec{e}_1 + \dots + a_n\vec{e}_n) = \vec{w}$$

$$\Rightarrow a_1 A\vec{e}_1 + \dots + a_n A\vec{e}_n = \vec{w}$$

$$\Rightarrow a_1\vec{c}_1 + \dots + a_n\vec{c}_n = \vec{w}$$

$$\Rightarrow \vec{w} \in CS(A)$$