

Section 2.5: Wronskians.

Example:

Are the functions $1+x^2$, $1-x^2$, $1+x$ linearly independent?

$$c_1(1+x^2) + c_2(1-x^2) + c_3(1+x) = 0$$

$$\Rightarrow c_1 + c_2 + c_3 = 0$$

$$c_3 = 0$$

$$c_1 - c_2 = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

\Rightarrow The only solution is $c_1 = c_2 = c_3 = 0$.

\Rightarrow linearly independent.

Example:

Are the functions $\{\sin^2(x), \cos^2(x), 5\}$ linearly independent?

$$c_1 \sin^2(x) + c_2 \cos^2(x) + 5c_3 = 0$$

$$\Rightarrow c_1(1 - \cos^2(x)) + c_2 \cos^2(x) + 5c_3 = 0$$

$$\Rightarrow (c_1 + 5c_3) + (c_2 - c_1)\cos^2(x) = 0$$

$$c_2 = \text{free variable}$$

$$c_1 = c_2$$

$$c_3 = -c_2/5$$

Linearly dependent.

Idea:

Set up n -equations by taking derivatives:

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$$

$$c_1 f_1'(x) + c_2 f_2'(x) + c_3 f_3'(x) = 0$$

$$c_1 f_1''(x) + c_2 f_2''(x) + c_3 f_3''(x) = 0$$

$$W(x) = \det \begin{pmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{pmatrix}$$

If $W(x) \neq 0$ for all $x \Rightarrow$ linearly independent.

Example:

$$f_1(x) = x$$

$$f_2(x) = \sin(x)$$

$$W(x) = \det \begin{pmatrix} x & \sin(x) \\ 1 & \cos(x) \end{pmatrix} = x \cos(x) - \sin(x) \neq 0.$$

\Rightarrow linearly independent.

Example:

$$f_1(x) = \sin^2(x), f_2(x) = \cos^2(x), f_3 = 5$$

$$W(x) = \det \begin{pmatrix} \sin^2(x) & \cos^2(x) & 5 \\ 2\sin(x)\cos(x) & -2\sin(x)\cos(x) & 0 \\ 2\cos^2(x) - 2\sin^2(x) & -2\cos^2(x) + 2\sin^2(x) & 0 \end{pmatrix}$$

$$= 0$$

\Rightarrow linearly dependent.

Example:

$$f_1 = 1, f_2 = e^x, f_3 = e^{2x}$$

$$W(x) = \det \begin{pmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{pmatrix}$$

$$= 2e^{3x} \neq 0$$

\Rightarrow linearly independent.