

Section 3.4: Linear Differential Equations.

$$p(x) y' + q_1(x) y(x) = q_2(x)$$

- Linear differential equation since first power in y and its derivatives.

- homogeneous if $q_2 = 0$

- inhomogeneous if $q_2 \neq 0$.

Equivalent form!

$$\boxed{\begin{aligned} y' + p(x)y(x) &= q(x). \\ y(x_0) &= y_0 \end{aligned}}$$

How do we solve??

Example:

$$1. \quad e^x y' + e^x y = x^2, \quad y(0) = y_0$$
$$\Rightarrow \frac{d}{dx}(e^x y) = x^2 \quad \text{or} \quad \int_{e^{y_0}}^{e^x} d(e^x y) = \int_0^x x^2 dx$$

$$\Rightarrow e^x y = \frac{x^3}{3} + c$$

$$\Rightarrow y e^x - y_0 = \frac{x^3}{3}$$

$$\Rightarrow y = \left(\frac{x^3}{3} + c \right) e^{-x}$$

$$y = \left(\frac{x^3}{3} + y_0 \right) e^{-x}$$

$$y(0) = y_0 \Rightarrow y_0 = c$$

$$y(x) = \left(\frac{x^3}{3} + y_0 \right) e^{-x}$$

$$2. y' = 2y + x$$

$$\Rightarrow y' - 2y = x$$

We would like to use the product rule. Multiply both sides by $e^{f(x)}$.

$$e^{f(x)} y' - 2e^{f(x)} y = e^{f(x)} x$$
$$\Rightarrow \frac{d}{dx} (e^{f(x)} y) = e^{f(x)} x \quad (*)$$

$$\Rightarrow e^{f(x)} y' + f'(x) e^{f(x)} y = e^{f(x)} x$$

$$f'(x) = -2$$

$$\Rightarrow f(x) = -2x$$

Therefore, substituting back into (*) we have that:

$$\frac{d}{dx} (e^{-2x} y) = e^{-2x} x$$

$$\Rightarrow e^{-2x} y = \int x e^{-2x} dx$$

$$= -\frac{x}{2} e^{-2x} + \int \frac{1}{2} e^{-2x} dx$$

$$= -\frac{x}{2} e^{-2x} - \frac{1}{4} e^{-2x} + C$$

$$\Rightarrow y = -\frac{x}{2} - \frac{1}{4} + C e^{2x}$$

$$3. \frac{dy}{dx} - \frac{1}{x} y = 0$$

$$e^{f(x)} \frac{dy}{dx} - \frac{1}{x} e^{f(x)} y = 0$$

$$f'(x) = -\frac{1}{x}$$

$$f(x) = -\ln(x)$$

Therefore, we obtain:

$$\frac{d}{dx} \left(e^{f(x)} y(x) \right) = 0$$

$$\Rightarrow \frac{d}{dx} \left(\frac{1}{x} y(x) \right) = 0$$

$$\Rightarrow \frac{1}{x} y(x) = C$$

$$\Rightarrow y(x) = Cx.$$

This is also a separable equation.

$$\frac{dy}{dx} = \frac{1}{x} y$$

$$\Rightarrow \int \frac{1}{y} dy = \int \frac{1}{x} dx$$

$$\Rightarrow \ln(y) = \ln(x) + C$$

$$y = Cx.$$

4. $y' = xy - x$

$$y(1) = 2$$

$$\Rightarrow \frac{dy}{dx} = x(y-1)$$

$$\Rightarrow \int_2^y \frac{1}{y-1} dy = \int_1^x x dx$$

$$\Rightarrow \ln(y-1) - \ln(1) = \frac{x^2}{2} - \frac{1}{2}$$

$$\Rightarrow y-1 = \exp\left(\frac{x^2-1}{2}\right)$$

$$\Rightarrow y = 1 + \exp\left(\frac{x^2-1}{2}\right).$$

We also have that:

$$y' - xy = -x$$
$$e^{f(x)} y' - x e^{f(x)} y = -x e^{f(x)}$$
$$f'(x) = -x$$
$$f(x) = \frac{-x^2}{2}$$

$$\Rightarrow \frac{d}{dx} \left(e^{-x^2/2} y \right) = -x e^{-x^2/2}$$

$$\int_{2e^{-1/2}}^{e^{-x^2/2}} d \left(e^{-x^2/2} y \right) = \int_1^x -x e^{-x^2/2} dx$$

$u = x^2/2 \Rightarrow du = x dx$

$$\Rightarrow e^{-x^2/2} y - 2e^{-1/2} = \int_{1/2}^{x^2/2} -e^{-u} du$$

$$\Rightarrow e^{-x^2/2} y - 2e^{-1/2} = e^{-x^2/2} - e^{-1/2}$$

$$\Rightarrow y = 1 + e^{1/2} e^{x^2/2}$$
$$\approx 1 + \exp\left(\frac{x^2-1}{2}\right)$$