Section 4.2: Homogeneous Constant Coefficient Systems.

Example:

$$y''' - 4y' = 0$$

Guess:

$$y = e^{\lambda x}$$

$$\lambda^3 - 4\lambda = 0$$

$$\Rightarrow \lambda = 0, \ \lambda = \pm 1$$

General Solution is

$$y(x) = c_1 e^{0x} + c_2 e^{x} + c_3 e^{-x}$$

$$= c_1 + c_2 e^{x} + c_3 e^{-x}.$$

Example:

$$y'' + 5y' + 6y = 0$$

$$y(0) = 2$$

$$y'(0) = 3$$

$$y = e^{\lambda x}$$

$$\Rightarrow \lambda^2 + 5\lambda + 6 = 0$$

$$(\lambda + 3)(\lambda + 2) = 0$$

$$\Rightarrow \lambda = -3, -2$$

$$y(x) = c_1 e^{-3x} + c_2 e^{-2x}$$

$$y(0) = c_1 + c_2 = 2$$

$$y'(0) = -3c_1 - 2c_2 = 3$$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ -3 & -2 & 3 \end{array}\right] +3R2 \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 9 \end{array}\right]$$

$$c_2 = 9 \qquad \Rightarrow y(x) = -7e^{-3x} + 9e^{-2x}.$$

$$c_1 = -7$$

Reduction of Order:

Consider the equation:

$$y'' + p(t)y' + q(t)y = 0$$

with solution $y_1(t)$. Let

$$y_2(t) = v(t)y_1(t)$$

$$\Rightarrow y_2'(t) = v'y_1 + vy_1'$$

$$y_2'' = v''y_1 + 2v'y_1' + vy_1''$$

$$\Rightarrow v''y_1 + 2v'y_1' + vy_1'' + p(t)(v'y_1 + vy_1') + q(t)v \cdot y_1 = 0$$

$$\Rightarrow v''y_1 + 2v'y_1' + v\underbrace{(y_1'' + p(t)y_1' + q(t)y_1)}_{=0} + p(t) \cdot v'y_1 = 0$$

$$\Rightarrow \quad y_1 v'' + (2y_1' + py_1)v' = 0$$

$$\downarrow$$

First order equation in v'

Example:

$$y'' - 2y' + y = 0$$

Guess

$$y = e^{\lambda x}$$

$$\Rightarrow \lambda^2 - 2x + 1 = 0$$

$$\Rightarrow (\lambda - 1)^2 = 0$$

$$\Rightarrow \lambda = 1$$

One solution is

$$y_1 = e^x$$

Guess another solution of the form

$$y_2 = v(x)e^x$$

$$\Rightarrow y_2' = v'e^x + ve^x$$

$$y_2'' = v''e^x + 2v'e^x + ve^x$$

$\Rightarrow v''e^x + 2v'e^x + \cancel{ve^x} - 2v'e^x - \cancel{2ve^x} + \cancel{ve^x}$

$\Rightarrow v''e^x = 0$

$\Rightarrow v(x) = cx + d$

General solution is

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$
$$= c_1 e^x + c_2 (c + dx) e^x$$
$$= c_1 e^x + c_2 x e^x.$$

Example:

Given $y_1 = t^{-1}$ is a solution of

$$2t^2 y'' + 3ty' - y = 0$$

find ~~a~~ second linear ind. solution.

$$y_2 = v t^{-1}$$

$$y_2' = \frac{-1}{t^2} v + v' t^{-1}$$

$$y_2'' = \frac{2}{t^3} v - \frac{1}{t^2} v' + v'' t^{-1} - \frac{v'}{t^2}$$

$$\Rightarrow \cancel{\frac{4}{t} v} - 4v' + v'' t - \cancel{\frac{3}{t} v} + 3v' - \cancel{\frac{v}{t}} = 0$$

$$t v'' - v' = 0$$

Let $\quad u = \dfrac{dv}{dt}$

$$\Rightarrow t \frac{du}{dt} = u$$

$$\Rightarrow \int \frac{1}{u} du = \int \frac{1}{t} dt$$

$$\ln(u) = \ln|t| + c$$

$\Rightarrow u = ct$

$\Rightarrow v = ct^2 + d$

Therefore,

$$y_2(t) = v \cdot \frac{1}{t}$$

$$= ct + \frac{d}{t}.$$

The two linearly dependent solutions are

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

$$= \frac{c_1}{t} + c_2 t.$$

Complex Numbers and Complex Roots

$z = x + yi$, $z \in \mathbb{C}$ --> complex numbers

↑ real # ↖ real # (vector space)

$i^2 = -1$

example:

$z_1 = 5 + 3i$, $z_2 = 2 - 2i$

$-\ z_1 \cdot z_2 = (5+3i)(2-2i)$

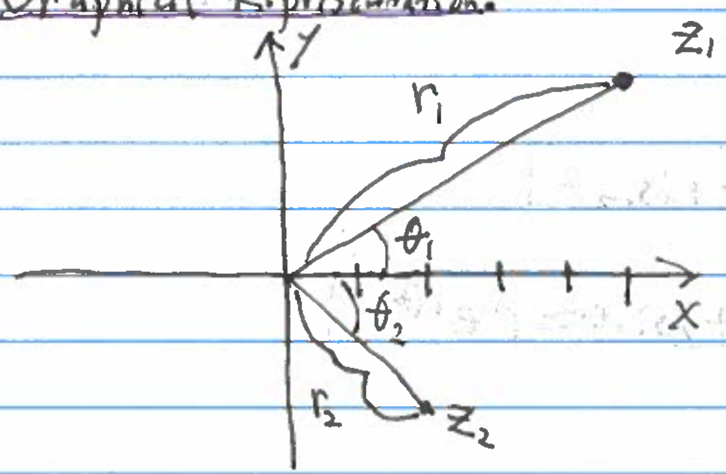
$= 10 + 6i - 10i + 6$

$= 16 - 4i$

$-\ z_1 + z_2 = 7 + i$

$-\ \dfrac{z_1}{z_2} = \dfrac{5+3i}{2-2i} \dfrac{(2+2i)}{(2+2i)} = \dfrac{10+14i-6}{8} = \dfrac{4+14i}{8} = \dfrac{1}{2} + \dfrac{7}{4}i$

Graphical Representation:



$r_1 = \sqrt{25+9} = \sqrt{34}$, $\tan\theta_1 = 3/5$

$r_2 = \sqrt{8} = 2\sqrt{2}$, $\tan\theta_2 = -2/2 \Rightarrow \theta_2 = -\pi/4$

$z_1 = r_1 \cos\theta_1 + r_1 \sin\theta_1 i$

$z_2 = r_2 \cos\theta_2 + r_2 \sin\theta_2 i$

Euler's Formula:

1. $e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{6} + \ldots$

$= 1 + i\theta - \frac{\theta^2}{2} - \frac{i\theta^3}{6} + \ldots$

2. $\cos(\theta) = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + \ldots$

3. $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots$

$\Rightarrow \cos\theta + i\sin\theta = 1 + i\theta - \frac{\theta^2}{2} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \ldots$

$= e^{i\theta}.$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

$$\boxed{z = x + iy = r\cos\theta + i\sin\theta = re^{i\theta}.}$$

Example:

1. $y''+4y=0$

Guess:

$y=e^{\lambda x}$

$y'=\lambda e^{\lambda x}$

$y''=\lambda^2 e^{\lambda x}$

$\Rightarrow \lambda^2+4=0$

$\Rightarrow \lambda=\pm 2i$

$y=c_1 e^{2ix}+c_2 e^{-2ix}$

$y=c_1(\cos(2x)+i\sin(2x))+c_2(\cos(2x)-i\sin(2x))$

$=c_1\cos(2x)+c_2\sin(2x).$

2. $2y''+8y'+26y=0$

$y''+4y'+13y=0$

Guess:

$y=e^{\lambda x}$

$\Rightarrow \lambda^2+4\lambda+13=0$

$\lambda=\dfrac{-4\pm\sqrt{16-4\cdot 13}}{2}$

$=\dfrac{-4\pm 2\sqrt{4-13}}{2}$

$=-2\pm 3i$

$y=c_1 e^{(-2+3i)x}+c_2 e^{(-2-3i)x}$

$=e^{-2x}(c_1 e^{3ix}+c_2 e^{-3ix})$

$=c_1 e^{-2x}\cos(3x)+c_2 e^{-2x}\sin(3x)$