Section 4.3: Method of Undetermined Coefficients

Example:

$y'' - 3y' - 4y = 3e^{2t}$ → Find the general solution.

$\underbrace{\left(\frac{d^2}{dt^2} - 3\frac{d}{dt} - 4\right)}_{L} y = 3e^{2t}$

$Ly = 3e^{2t}$

Homogeneous solution:

$y_g = e^{\lambda t}$

$\Rightarrow \lambda^2 - 3\lambda - 4 = 0$

$(\lambda - 4)(\lambda + 1) = 0$

$\lambda = 4, -1$

$y_H = c_1 e^{4t} + c_2 e^{-t}$

Particular solution:

$y_p = Ae^{2t}$

$\Rightarrow 4A - 6A - 4A = 3$

$A = -1/2$

General Solution:

$y = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{2}e^{2t}$.

Example:

$$y'' - 3y' - 4y = -8e^{t}\cos(2t)$$

$$y_p = Ae^{t}\cos(2t) + Be^{t}\sin(2t)$$

$$y_p' = Ae^{t}\cos(2t) - 2Ae^{t}\sin(2t) + Be^{t}\sin(2t) + 2Be^{t}\cos(2t)$$

$$y_p'' = Ae^{t}\cos(2t) - 2Ae^{t}\sin(2t) - 2Ae^{t}\sin(2t) - 4Ae^{t}\cos(2t)$$
$$+ Be^{t}\sin(2t) + 2Be^{t}\cos(2t) + 2Be^{t}\cos(2t) - 4Be^{t}\sin(2t)$$

$$= (-3A + 4B)\cos(2t) + (-4A - 3B)\sin(2t) = -8e^{t}\cos(2t)$$

$$\begin{aligned} -3A + 4B &= -8 \\ -4A - 3B &= 0 \end{aligned} \Rightarrow \left[\begin{array}{cc|c} -3 & 4 & -8 \\ -4 & -3 & 0 \end{array}\right] \begin{array}{l} \\ -R1 \end{array} \Rightarrow \left[\begin{array}{cc|c} -3 & 4 & -8 \\ +1 & +7 & 0 \end{array}\right] \begin{array}{l} +3R1 \\ \\ \end{array}$$

$$\Rightarrow \left[\begin{array}{cc|c} 0 & 11 & -8 \\ 1 & 7 & 0 \end{array}\right]$$

Example:

$y'' + 9y = t^2 e^{3t} + 6$

Homogeneous Solution:

$y = c_1 \cos(3t) + c_2 \sin(3t)$

Particular Solution:

$y_p = At^2 e^{3t} + Bte^{3t} + Ce^{3t} + D$

$y_p' = 2Ate^{3t} + 3At^2 e^{3t} + Be^{3t} + 3Bte^{3t} + 3Ce^{3t}$

$y_p'' = 6Ate^{3t} + 2Ae^{3t} + 6Ate^{3t} + 9At^2 e^{3t} + 3Be^{3t}$
$\quad + 3Be^{3t} + 9Bte^{3t} + 9Ce^{3t}$

$t^2 e^{3t}$:
$9A = 1$

te^{3t}:
$12A + 9B + 9B = 0$

e^{3t}:
$2A + 6B + 9C + 9C = 0$

1:
$9D = 6$

$$\left[\begin{array}{ccc|c} 9 & 0 & 0 & 1 \\ 12 & 12 & 0 & 0 \\ 2 & 6 & 18 & 0 \end{array}\right]$$

$A = 1/9$

$\rightarrow 4/3 + 12B = 0$

$B = -1/9$

$\Rightarrow 2/9 - 6/9 + 18C = 0$

$\Rightarrow -4/9 = -18C$

$C = -$