

## Section 5.1: Linear Transformations

$$f: X \rightarrow Y$$

is linear transformation if

$$- f(u+v) = f(u) + f(v)$$

$$- f(cu) = cf(u)$$

Notation -

$$- X = \text{domain} \quad ; \quad \bar{\quad}$$

$$- Y = \text{codomain}$$

$$- \{f(x) : x \in X\} = \text{range of } f.$$

Example:

1.  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  defined by:

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y-z \\ x+2y+z \end{bmatrix}$$

$$\Rightarrow T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

is linear since

$$A(u+v) = Au + Av$$

$$A(cu) = cAu.$$

2.  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^2 \\ x+y \end{bmatrix}$$

is not linear since

$$T \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + T \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1^2 + x_2^2 \\ x_1 + x_2 + y_1 + y_2 \end{bmatrix}$$

$$T \begin{bmatrix} (x_1+x_2) \\ (y_1+y_2) \end{bmatrix} = \begin{bmatrix} (x_1+x_2)^2 \\ x_1+x_2+y_1+y_2 \end{bmatrix} = \begin{bmatrix} x_1^2 + 2x_1x_2 + x_2^2 \\ x_1+x_2+y_1+y_2 \end{bmatrix}$$

$$3. T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+1 \\ y \end{bmatrix}$$

is not linear since

$$T \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix} = \begin{bmatrix} x_1+x_2+1 \\ y_1+y_2 \end{bmatrix}$$

$$T \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + T \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1+1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2+1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1+x_2+2 \\ y_1+y_2 \end{bmatrix}$$

4.  $\frac{d}{dx}$  is a linear operator

5.  $\int_a^b f(x) dx$  is a linear operator.

Example:

1. Suppose  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  satisfies

$$T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \quad T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \quad T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

What is  $T \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ ?

$$\begin{aligned} T \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} &= T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 11 \\ 14 \end{bmatrix} \end{aligned}$$

(You just need to know the action on a basis!)

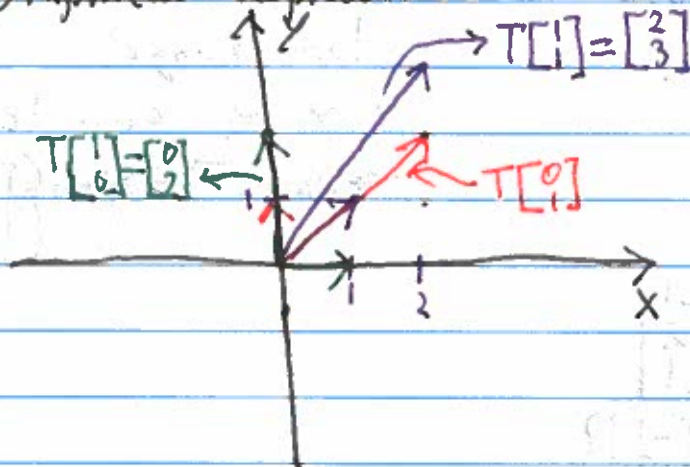
2. Suppose  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  satisfies

$$T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

What is  $T \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ?

$$\begin{aligned}
 \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
 \Rightarrow T \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= T \begin{bmatrix} 1 \\ 1 \end{bmatrix} - T \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
 &= \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\
 &= \begin{bmatrix} 2 \\ 1 \end{bmatrix}
 \end{aligned}$$

Graphical Representation.



Def

- $\text{Ker}(T) = \{v \in V : Tv = 0\}$
- $\text{ran}(T) = \{Tv : v \in \text{Dom}(T)\}$

Example:

Find the kernel and range of  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  defined by:

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

If  $\vec{x} \in \text{ker}(T)$  then +

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & -1 & | & 0 \\ 1 & 2 & 1 & | & 0 \end{bmatrix}$$

If  $y \in \text{range}(T)$  then there exists  $\vec{x}$  such that

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & -1 & | & y_1 \\ 1 & 2 & 1 & | & y_2 \end{bmatrix}$$

Row reduce all at once:

$$\begin{bmatrix} 1 & 1 & -1 & 0 & y_1 \\ 1 & 2 & 1 & 0 & y_2 \end{bmatrix} \sim R1$$
$$\Rightarrow \begin{bmatrix} 1 & 1 & -1 & 0 & y_1 \\ 0 & -1 & 2 & 0 & y_2 - y_1 \end{bmatrix}$$

For kernel:

$$y = 2z$$

$$x + y - z = 0$$

$$x = -z$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ 2z \\ z \end{bmatrix} = z \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{Ker}(T) = \text{span} \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

For range:

$y_1, y_2$  can be anything

$$\Rightarrow \text{ran}(T) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

## Theorems

$T: V \rightarrow W$  is a linear transformation

1.  $\dim(\text{ker}(T)) + \dim(\text{ran}(T)) = \dim V$

2.  $\dim(\text{RS}(A)) = \dim(\text{CS}(A))$