

Section 5.3: Matrices for Linear Transformations

$$\begin{aligned} V = \mathbb{R}^n &\rightarrow \text{dimension } n \\ V = \mathbb{P}_n &\rightarrow \text{dimension } n+1 \end{aligned} \quad \left. \vphantom{\begin{aligned} V = \mathbb{R}^n \\ V = \mathbb{P}_n \end{aligned}} \right\} \text{finite dimensional}$$

Linear transformations between finite dimensional vector spaces can always be represented by matrices.

Example:

1. $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

$$T(\vec{e}_2) = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix},$$

$$T(\vec{e}_3) = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}.$$

Let $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Therefore,

$$\begin{aligned} T(v) &= T(x\vec{e}_1 + y\vec{e}_2 + z\vec{e}_3) \\ &= xT(\vec{e}_1) + yT(\vec{e}_2) + zT(\vec{e}_3) \\ &= x \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + z \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \end{aligned}$$

$$= (x+2y+3z)\vec{e}_1 + (2x+3y+1z)\vec{e}_2 + (3x+y+2z)\vec{e}_3$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$[T]_{\mathcal{B}}^{\mathcal{B}} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

2. $T: P_2 \rightarrow P_2$ defined by

$$T(ax^2 + bx + c) = 2ax + b.$$

Standard basis: $\{x^2, x, 1\}$.

$$T(x^2) = 2x, \quad T(x) = 1, \quad T(1) = 0$$

$$T(v) = T(ax^2 + bx + c)$$

$$= aT(x^2) + bT(x) + T(c)$$

$$= 2ax + b$$

$$= 2a\vec{e}_2 + b\vec{e}_1$$

$$[T]_{\alpha}^{\alpha} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Check! $\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 2a \\ b \end{bmatrix}$

Matrix Representation

↑

→ Matrix Representation

Example:

$$T(x, y, z) = (5x + y, 3x + 2y)$$

$$\alpha = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \quad \beta = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

Find

$$[T]_{\alpha}^{\alpha}$$

$$[T]_{\beta}^{\beta}$$

$$[T]_{\alpha}^{\beta}$$

$$[T]_{\beta}^{\alpha}$$

$$T \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}_{\alpha}$$

$$\Rightarrow [T]_{\alpha}^{\alpha} = \begin{bmatrix} 5 & 1 \\ 3 & 2 \end{bmatrix}$$

$$T \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\alpha}$$

Check: $[T]_{\alpha}^{\alpha} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5x + y \\ 3x + 2y \end{bmatrix}$

What about $[T]_{\beta}^{\beta}$?

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\alpha}\right) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\beta}\right) = \begin{bmatrix} 6 \\ 5 \end{bmatrix}_{\alpha}$$

$$T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}_{\alpha}\right) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\beta}\right) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}_{\alpha}$$

$$\Rightarrow c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}, \quad c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc|cc} 1 & 1 & 6 & 4 \\ 1 & -1 & 5 & 1 \end{array} \right] \xrightarrow{-R1} \left[\begin{array}{cc|cc} 1 & 1 & 6 & 4 \\ 0 & -2 & -1 & -3 \end{array} \right] \cdot 1/2$$

$$\Rightarrow \left[\begin{array}{cc|cc} 1 & 1 & 6 & 4 \\ 0 & 1 & 1/2 & 3/2 \end{array} \right] \xrightarrow{-R2} \left[\begin{array}{cc|cc} 1 & 0 & 11/2 & 5/2 \\ 0 & 1 & 1/2 & 3/2 \end{array} \right]$$

Therefore,

$$\begin{bmatrix} 6 \\ 5 \end{bmatrix}_{\alpha} = \begin{bmatrix} 11/2 \\ 1/2 \end{bmatrix}_{\beta}, \quad \begin{bmatrix} 4 \\ 1 \end{bmatrix}_{\alpha} = \begin{bmatrix} 5/2 \\ 3/2 \end{bmatrix}_{\beta}$$

Consequently,

$$\boxed{[T]_{\beta}^{\beta} = \begin{bmatrix} 11/2 & 5/2 \\ 1/2 & 3/2 \end{bmatrix}}$$

Check:

$$[T]_{\beta}^{\beta} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\beta} = \begin{bmatrix} 11/2 \\ 1/2 \end{bmatrix}_{\beta} = 11/2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\alpha} + 1/2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{\alpha} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}_{\alpha}$$

What about $[T]_{\alpha}^{\beta}$?

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\alpha}\right) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}_{\alpha}, \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\alpha}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\alpha}$$

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \quad c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc|cc} 1 & 1 & 5 & 1 \\ 1 & -1 & 3 & 2 \end{array} \right] \xrightarrow{-R_1} \left[\begin{array}{cc|cc} 1 & 1 & 5 & 1 \\ 0 & -2 & -2 & 1 \end{array} \right] \xrightarrow{/2} \left[\begin{array}{cc|cc} 1 & 1 & 5 & 1 \\ 0 & 1 & 1 & -\frac{1}{2} \end{array} \right] \xrightarrow{-R_2}$$

$$\Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 4 & \frac{1}{2} \\ 0 & 1 & 1 & -\frac{1}{2} \end{array} \right]$$

$$\Rightarrow \begin{bmatrix} 5 \\ 3 \end{bmatrix}_{\alpha} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}_{\beta}, \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\alpha} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}_{\beta}$$

$$\Rightarrow [T]_{\alpha}^{\beta} = \begin{bmatrix} 4 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}$$

Check!

$$[T]_{\alpha}^{\beta} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\alpha} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}_{\beta} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\alpha} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{\alpha} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}_{\alpha}$$

What about $[T]_{\beta}^{\alpha}$?

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\alpha}\right) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\beta}\right) = \begin{bmatrix} 6 \\ 5 \end{bmatrix}_{\alpha}, \quad T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}_{\alpha}\right) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\beta}\right) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}_{\alpha}$$

$$\Rightarrow [T]_{\beta}^{\alpha} = \begin{bmatrix} 6 & 4 \\ 5 & 1 \end{bmatrix}$$

Change of Basis

$$\alpha = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Convert:

$$\vec{v}_\alpha = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_\alpha \text{ into } \beta \text{ representation, } \vec{v}_\beta$$

$$\vec{v}_\beta = [I]_\alpha^\beta \vec{v}_\alpha \quad \text{and} \quad [I]_\beta^\alpha \vec{v}_\beta = \vec{v}_\alpha$$

$$\Rightarrow [I]_\beta^\alpha = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \Rightarrow [I]_\alpha^\beta = ([I]_\beta^\alpha)^{-1}$$

$$\begin{bmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 1 & -1 & 1 & | & 0 & 1 & 0 \\ 2 & 1 & 1 & | & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\substack{-R1 \\ -2R1}} \begin{bmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & -2 & 0 & | & -1 & 1 & 0 \\ 0 & -1 & -1 & | & -2 & 0 & 1 \end{bmatrix} \xrightarrow{\substack{-1/2 \\ -1}}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & 1/2 & -1/2 & 0 \\ 0 & 1 & 1 & | & 2 & 0 & -1 \end{bmatrix} \xrightarrow{\substack{-R2 \\ -R2}} \begin{bmatrix} 1 & 0 & 1 & | & 1/2 & 0 & 0 \\ 0 & 1 & 0 & | & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & | & 3/2 & 1/2 & -1 \end{bmatrix} \xrightarrow{-R3}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & | & -1 & 0 & 0 \\ 0 & 1 & 0 & | & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & | & 3/2 & 1/2 & -1 \end{bmatrix}$$

Therefore,

$$[I]_\alpha^\beta = P = \begin{bmatrix} -1 & 0 & 1 \\ 1/2 & -1/2 & 0 \\ 3/2 & 1/2 & -1 \end{bmatrix}, \quad [I]_\beta^\alpha = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

Columns are basis vectors.

$$[I]_\alpha^\beta \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_\alpha = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_\beta$$

Check!

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_\beta = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_\alpha$$

Change of Basis for matrix:
Convert $[T]_{\alpha}^{\alpha}$ into $[T]_{\beta}^{\beta}$

$$[T]_{\beta}^{\beta} \vec{v}_{\beta} = \vec{w}_{\beta}$$
$$\Rightarrow [T]_{\beta}^{\beta} [I]_{\alpha}^{\beta} \vec{v}_{\alpha} = [I]_{\alpha}^{\beta} \vec{w}_{\alpha}$$

$$\Rightarrow ([I]_{\alpha}^{\beta})^{-1} [T]_{\beta}^{\beta} [I]_{\alpha}^{\beta} \vec{v}_{\alpha} = \vec{w}_{\alpha}$$

$$\Rightarrow [T]_{\alpha}^{\alpha} = ([I]_{\alpha}^{\beta})^{-1} [T]_{\beta}^{\beta} [I]_{\alpha}^{\beta}$$

$$[T]_{\beta}^{\beta} = ([I]_{\alpha}^{\beta}) [T]_{\alpha}^{\alpha} ([I]_{\alpha}^{\beta})^{-1}$$

$$[T]_{\beta}^{\beta} = ([I]_{\beta}^{\alpha})^{-1} [T]_{\alpha}^{\alpha} [I]_{\beta}^{\alpha}$$

$$\boxed{[T]_{\beta}^{\beta} = P^{-1} [T]_{\alpha}^{\alpha} P}$$

Recall:

P is the matrix with columns given by the basis β .