Lecture 20: Eigenvalues and Eigenvectors

Diagonal Matrices:

$$\begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} = A = \left[A\vec{e}_1 \middle| \cdots \middle| A\vec{e}_n \right] = \left[d_1\vec{e}_1 \middle| \cdots \middle| d_n\vec{e}_n \right]$$

$A\vec{e}_i = d_i\vec{e}_i \rightarrow$ Similar to matrix multiplication.

Eigenvalues and Eigenvectors

- A – $n \times n$ matrix
- $\vec{v}$ is an eigenvector with corresponding eigenvalue if

$$A\vec{v} = \lambda\vec{v}$$

Example:

Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}$$

If $\vec{v}$ is an eigenvector with eigenvalue λ then.

$$A\vec{v} = \lambda\vec{v}$$

$$\Rightarrow (A - \lambda I)\vec{v} = 0.$$

This has a nontrivial solution if:

$$\det\left(\begin{bmatrix} 1-\lambda & -3 \\ -2 & 2-\lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda) - 6 = 0$$

$$\lambda^2 - 3\lambda + 2 - 6 = 0$$

$$\lambda^2 - 3\lambda - 4 = 0$$

$$(\lambda - 4)(\lambda + 1) = 0$$

$\Rightarrow \lambda_1 = 4$ and $\lambda_2 = -1$ are eigenvalues.

How do we find eigenvectors?

1. For λ_1:

$$A-\lambda_1 I=\begin{bmatrix}-3 & -3\\ -2 & -2\end{bmatrix}$$

$$\Rightarrow (A-\lambda_1 I)\vec{v}_1=0$$

$$\Rightarrow \vec{v}_1\in NS(A-\lambda_1 I)=E_{\lambda_1}$$

eigenspace of $\lambda_1=4$

$$\begin{bmatrix}-3 & -3\\ -2 & -2\end{bmatrix}\rightarrow\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{matrix}\\ -R1\end{matrix}\rightarrow\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$$

$$\Rightarrow x=-y$$

$$\Rightarrow \vec{v}_1=\begin{bmatrix}-y\\ y\end{bmatrix}=c_1\begin{bmatrix}-1\\ 1\end{bmatrix}$$

2. For λ_2:

$$A-\lambda_2 I=\begin{bmatrix}2 & -3\\ -2 & 3\end{bmatrix}\begin{matrix}\\ \rightarrow +R1\end{matrix}\rightarrow\begin{bmatrix}2 & -3\\ 0 & 0\end{bmatrix}$$

$$\rightarrow x=\tfrac{3}{2}y$$

$$\vec{v}_2=\begin{bmatrix}\tfrac{3}{2}y\\ y\end{bmatrix}=c_1\begin{bmatrix}\tfrac{3}{2}\\ 1\end{bmatrix}$$

The eigenvalues are $\lambda_1=4$ and $\lambda_2=-1$ with associated eigenvectors

$$\vec{v}_1=\begin{bmatrix}-1\\ 1\end{bmatrix},\quad \vec{v}_2=\begin{bmatrix}3\\ 2\end{bmatrix}$$

The eigenspaces are

$$E_{-4}=\text{span}\left\{\begin{bmatrix}-1\\ 1\end{bmatrix}\right\},\quad E_{1}=\text{span}\left\{\begin{bmatrix}3\\ 2\end{bmatrix}\right\}$$

For this problem define

$$\alpha = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \quad \beta = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$$

$$[T]_\alpha^\alpha = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}$$

$$[I]_\beta^\alpha = \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix} \longrightarrow \left[\begin{array}{cc|cc} -1 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \updownarrow$$

$$\longrightarrow \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ -1 & 3 & 1 & 0 \end{array}\right] \begin{array}{l} \\ +R1 \end{array}$$

$$\longrightarrow \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 5 & 1 & 1 \end{array}\right] \begin{array}{l} \\ /5 \end{array} \longrightarrow \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & 1/5 & 1/5 \end{array}\right] \begin{array}{l} -2R2 \\ \\ \end{array}$$

$$\longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & -2/5 & 3/5 \\ 0 & 1 & 1/5 & 1/5 \end{array}\right]$$

$$[I]_\alpha^\beta = \left([I]_\beta^\alpha\right)^{-1} = \begin{bmatrix} -2/5 & 3/5 \\ 1/5 & 1/5 \end{bmatrix}$$

$[T]_\alpha^\alpha \vec{V}_\alpha = \vec{W}_\alpha \longrightarrow$ Normal Matrix multiplication.

$$[T]_\alpha^\alpha [I]_\beta^\alpha \vec{V}_\beta = [I]_\beta^\alpha \vec{W}_\beta.$$

$$\Rightarrow \left([I]_\beta^\alpha\right)^{-1} [T]_\alpha^\alpha [I]_\beta^\alpha \vec{V}_\beta = \vec{W}_\beta = [T]_\beta^\beta \vec{V}_\beta.$$

$$\Rightarrow \begin{bmatrix} -2/5 & 3/5 \\ 1/5 & 1/5 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix}$$

Example:

Find the eigenvalues and eigenspaces of

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

1. Eigenvalues:

$\lambda_1 = 2, \ \lambda_2 = -1 \qquad , \ A\vec{v}_\lambda = \lambda\vec{v}$

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 2-\lambda & -1 & 3 \\ 0 & -1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{bmatrix}\right) = (2-\lambda)(\lambda+1)^2$$

2. Eigenspaces:

a.) $\lambda_1 = 2$

$$(A - \lambda_1 I) = \begin{bmatrix} 0 & -1 & 3 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \Rightarrow NS(A - \lambda_1 I) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$$

b.) $\lambda_1 = -1$

$$(A - \lambda_2 I) = \begin{bmatrix} 3 & -1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If $\vec{v} \in NS(A - \lambda_2 I)$ then

$$\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y/3 - z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} 1/3 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Eigenspaces:

$$E_{\lambda_1} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}, \ E_{\lambda_2} = \text{span}\left\{\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix}\right\}$$

Example:

Determine the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

1. $\det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix}\right)$

$$= 1 - 2\lambda + \lambda^2 + 1 = 0$$

$$= \lambda^2 - 2\lambda + 2 = 0$$

$$\Rightarrow \lambda = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i.$$

$\lambda_1 = 1+i, \quad \lambda_2 = 1-i = \bar{\lambda}_1 \rightarrow$ complex conjugate.

2. $A - \lambda_1 I = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1/i \\ 1 & -i \end{bmatrix} = \begin{bmatrix} 1 & -i \\ 1 & -i \end{bmatrix} -R1$

$$= \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$

$$NS(A - \lambda_1 I) = \text{span}\left\{\begin{bmatrix} i \\ 1 \end{bmatrix}\right\}$$

$$NS(A - \lambda_2 I) = \text{span}\left\{\begin{bmatrix} -i \\ 1 \end{bmatrix}\right\}$$