Lecture 21: Diagonalization and Jordan Canonical Form.

Suppose $A \in \mathbb{R}^{n \times n}$ with linearly independent eigenvectors with eigenvalues $\lambda_1, \ldots, \lambda_n$.

$$V = [\vec{v}_1 | \cdots | \vec{v}_n]$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$A\vec{v}_i = \lambda_i \vec{v}_i = \vec{v}_i(\lambda_i)$$

$$\Rightarrow A \cdot V = V\Lambda$$

$$\Downarrow$$

$$\underbrace{A}_{\text{standard representation}} = V\Lambda V^{-1} \quad \text{and} \quad \underbrace{\overset{[A]_\beta^\beta}{\Lambda}}_{\text{representation in basis of eigenvectors.}} = V^{-1}AV$$

A is diagonalizable if and only if
$$\dim(E_{\lambda_1}) + \cdots + \dim(E_{\lambda_m}) = n.$$

Example:

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}$$

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda-1 & 3 \\ 2 & \lambda-2 \end{bmatrix}\right)$$

From last time:

$$\lambda_1 = 4, \quad \lambda_2 = -1$$

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \underbrace{\begin{bmatrix} -1 & 3/2 \\ 1 & 1 \end{bmatrix}}_{P} = \underbrace{\begin{bmatrix} -1 & 3/2 \\ 1 & 1 \end{bmatrix}}_{P^{-1}} \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} -1 & 3/2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \updownarrow \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ -1 & 3/2 & 1 & 0 \end{array}\right] +R1 \Rightarrow \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 5/2 & 1 & 1 \end{array}\right] -\tfrac{2}{5}R2$$

$$\Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -5/2 & -3/2 \\ 0 & 5/2 & 1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -5/2 & -3/2 \\ 0 & 1 & 2/5 & 2/5 \end{array}\right]$$

$$\begin{bmatrix} -5/2 & -3/2 \\ 2/5 & 2/5 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 3/2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$$

Jordan Canonical Form

If $B = P^{-1}AP$ we say A and B are similar.

→ If A has n linearly ind. eigenvectors it is similar to a diagonal matrix.

→ If A has less than n linearly ind. eigenvectors A is similar to Jordan matrix.

Example:

$\det(\lambda I - A) = (\lambda-4)(\lambda-2)^3$ what are the possible Jordan canonical forms?

$$A \sim \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ or } A \sim \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$