Lecture 22: Homogeneous Systems

Example:

$$\dot{x} = -3x \qquad \Rightarrow \frac{d\vec{x}}{dt} = \begin{bmatrix} -3 & 6 \\ 0 & 4 \end{bmatrix}\vec{x}, \quad \vec{x} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$
$$\dot{y} = 4y$$

$$\frac{d\vec{x}}{dt} = A\vec{x} \rightarrow \text{Linear System}$$

If $\vec{x}_1, \vec{x}_2$ are solutions than so is

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$$

$$\frac{d\vec{x}}{dt} = c_1\frac{d\vec{x}_1}{dt} + c_2\frac{d\vec{x}_2}{dt} = c_1 A\vec{x}_1 + c_2 A\vec{x}_2 = A(c_1\vec{x}_1 + c_2\vec{x}_2)$$
$$= A\vec{x}$$

$$\frac{dx}{dt} = -3x \Rightarrow x(t) = c_1 e^{-3t}$$

$$\frac{dy}{dt} = 4y \Rightarrow y(t) = c_2 e^{4t}$$

$$\Rightarrow \vec{x} = c_1\begin{bmatrix} e^{-3t} \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ e^{4t} \end{bmatrix}.$$

Example:

$$\frac{dx}{dt} = A\vec{x}$$

If λ_1 is an eigenvalue with corresponding eigenvector $\vec{v}_1$ it follows that

$$\vec{x}_1 = e^{\lambda t}\vec{v}_1$$

is a solution.

$$\frac{d\vec{x}_1}{dt} = \lambda e^{\lambda t}\vec{v}_1 = e^{\lambda t}\lambda\vec{v}_1 = e^{\lambda t}A\vec{v}_1 = A(e^{\lambda t}\vec{v}_1) = A\vec{x}_1.$$

If there are n linearly ind. eigenvectors with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ the general solution is

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + \ldots + c_n e^{\lambda_n t}\vec{v}_n.$$

Example:

$\dot{\vec{x}} = A\vec{x}$ where $A = \begin{bmatrix} -1 & -2 \\ 0 & -2 \end{bmatrix}$. The eigenvalues are

$\lambda_1 = -1, \ \lambda_2 = -2$

If $\lambda_1 = -1$:

$$A - \lambda_1 I = \begin{bmatrix} 0 & -2 \\ 0 & -1 \end{bmatrix} \Rightarrow \text{eigenvector} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

If $\lambda_2 = -2$

$$A - \lambda_2 I = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow \text{eigenvector} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The general solution is therefore,

$$y = c_1 e^{-t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} + c_2 e^{-2t} \\ c_2 e^{-2t} \end{bmatrix}$$

Example:

$\dot{\vec{x}} = A\vec{x}$.

$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.