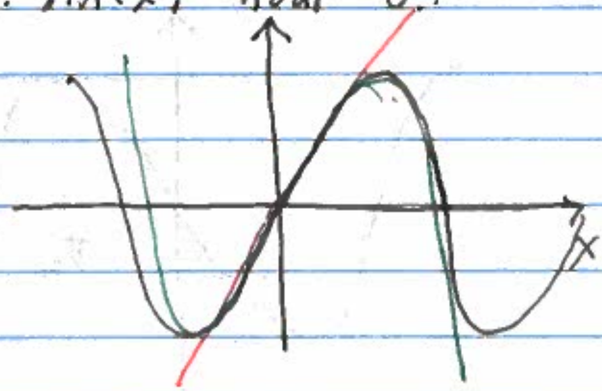
Lecture 23: Power Series Solutions.

How do you approximate a function by a polynomial?

1. $\sin(x)$ near 0:



Approximate by a tangent line.

a.) $y = mx + b$

$m = f'(0) = \cos(0) = 1$

$b = y\text{int} = 0.$

$\Rightarrow y_1 = x \rightsquigarrow$ best linear approximation.

b.) What about best quadratic approximation?

$y = c_0 + c_1 x + c_2 x^2$

$y(0) = f(0) \Rightarrow c_0 = f(0) = 0$

$y'(0) = f'(0) \rightarrow c_1 = f'(0) = 1$

$y''(0) = f''(0) \Rightarrow 2c_2 = f''(0) = 0$

$\Rightarrow y_2 = x \rightsquigarrow$ best quadratic approximation

c.) What about best cubic approximation?

$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3$

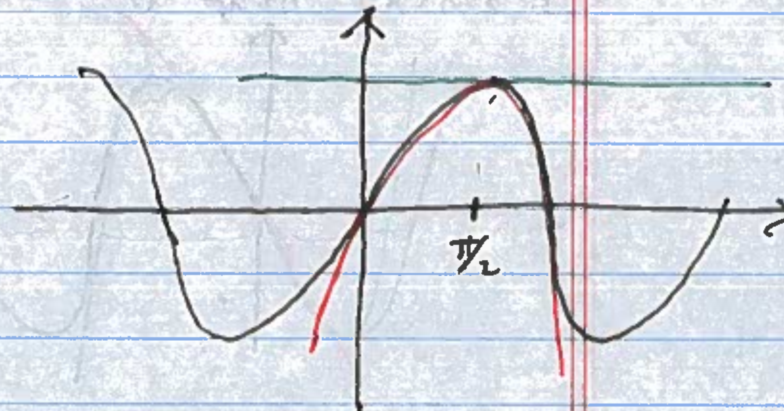
$y'''(0) = f'''(0) \Rightarrow 6c_3 = f'''(0) = -1$

$\Rightarrow y_3 = x - \frac{1}{6}x^3 \sim$ best cubic approximation.

2. $\sin(x)$ near $x=\pi/2$

$$\sin(x) = c_0 + c_1(x-\pi/2) + c_2(x-\pi/2)^2 + c_3(x-\pi/2)^3 + \dots$$

- $\sin(\pi/2) = 1 \Rightarrow c_0 = 1$
- $\sin'(x) = \cos(x)$
  $\Rightarrow \cos(\pi/2) = 0 = c_1$
- $\sin''(x) = -\sin(x)$
  $\Rightarrow -\sin(\pi/2) = -1 = 2c_2$
  $\Rightarrow c_2 = -1/2$



$$\sin(x) \approx \underbrace{1 - \tfrac{1}{2}(x-\pi/2)^2}_{\text{best quadratic approximation}}$$

<u>Power Series Appoximation of a function</u>

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

x_0 ↑ center of approximation.

Does this function make sense??

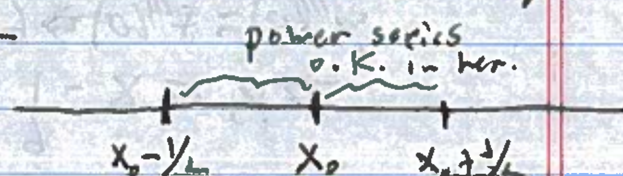
<u>Ratio Test:</u>

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\frac{(x-x_0)^{n+1}}{(x-x_0)^n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| |x-x_0| < 1$$

Let $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$. Therefore,

$$|x-x_0| < 1/L \quad 1/L = \text{radius of convergence.}$$

$$\Rightarrow x_0 - 1/L < x < x_0 + 1/L$$



Example:

- $f(x) = \frac{1}{1+x} = (1+x)^{-1}$; $f'(x) = -(1+x)^{-2}$, $f''(x) = 2(1+x)^{-3}$, ...

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$
$$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \dots$$
$$f''(x) = 2c_2 + 6c_3 x + \dots$$
$$\Downarrow$$
$$f(0) = 1 = c_0$$
$$f'(0) = -1 = c_1$$
$$f''(0) = 2 = 2c_2 \Rightarrow c_2 = 1$$
$$f(x) = 1 - x + x^2 - x^3 + \dots$$

Radius of convergence?

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1 = L$$

$\Rightarrow \rho = 1$.

$\Rightarrow \frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$ if $-1 < x < 1$

- $f(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$ if $-1 < x < 1$.

Important Taylor Series:

- $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$, $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$
- $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$, $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$
- $e^x = 1 + x + \frac{x^2}{2!} + \dots$, $\ln(1+x) = -1 + 2x - 3x^2 + \dots$

Example:

Solve:

$$y' = (x-1)^2 y \simeq (x-1)^2 y$$

$$y(1) = -1$$

Guess

$$y(x) = c_0 + c_1(x+1) + c_2(x+1)^2 + c_3(x+1)^3 + \dots$$

$$y'(x) = c_1 + 2c_2(x+1) + 3c_3(x+1)^2 + \dots$$

$$\Rightarrow (c_1 + 2c_2(x+1) + 3c_3(x+1)^2 + \dots) = (x-1)^2(c_0 + c_1(x+1) + c_2(x+1)^2 + c_3(x+1)^3 + \dots)$$

Gather powers:

- $(x+1)^0$: $c_1 = 0$
- $(x+1)^1$: $2c_2 = 0$
- $(x+1)^2$: $3c_3 = c_0$
- $(x+1)^3$: $4c_4 = c_1$
- $(x+1)^4$: $5c_5 = c_2$

Also:

$y(1) = -1$

$\Rightarrow c_0 = -1$

$\Rightarrow c_1 = 0$

$\Rightarrow c_2 = 0$

$\Rightarrow c_3 = -1/3$

$\Rightarrow c_4 = 0$

$\Rightarrow c_5 = 0$

$\Rightarrow c_6 = -1/3 \cdot 6 = -\frac{1}{3}\frac{1}{1\cdot 2}$

$c_9 = -1/3 \cdot \frac{1}{1\cdot 2\cdot 3}$

$c_{12} = -1/3 \cdot \frac{1}{1\cdot 2\cdot 3\cdot 4}$

$$\Rightarrow y(x) = -1 - 1/3\, x^3 - 1/3 \cdot \tfrac{1}{2!} x^6 - 1/3 \cdot 1/3!\, x^9 - 1/3 \cdot 1/3!\, x^{12} + \dots$$

$$= -1 - 1/3\,(x^3 + 1/2!\, x^6 + 1/3! + \dots)$$

$$= -1 - 1/3\,(e^{x^3} - 1)$$

$$= -2/3 - 1/3\, e^{x^3}.$$