

Section 1.3: Inverses of Matrices

Example:

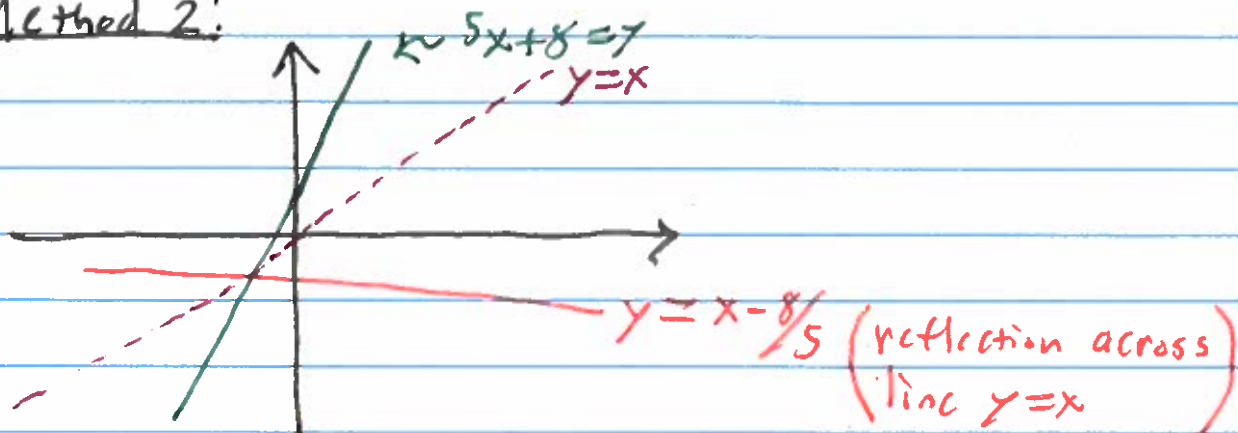
If $f(x) = 5x + 8$, what is $f^{-1}(x)$?

- Method 1:

$$f(x) = \text{five times } x + 8 = 5x + 8$$

$$f^{-1}(x) = x - 8 \text{ divided by } 5 = \frac{x - 8}{5}$$

- Method 2:



- Method 3:

We know:

$$f(f^{-1}(x)) = x$$

$$\Rightarrow 5f^{-1}(x) + 8 = x$$

$$\Rightarrow f^{-1}(x) = \frac{x - 8}{5}$$

→ Just solve for the inverse!!

Definition - If A, B are square matrices so that
 $AB = BA = I$,

then A is invertible and $A^{-1} = B$. If no such B exists then A is called singular.

Example:

- If $x, y \in \mathbb{R}$ and $xy = 0$ then $x = 0$ or $y = 0$.

- If $A, B \in \mathbb{R}^{n \times n}$ does $AB = 0$ imply $A = 0$ or $B = 0$??

- If $A, B \in \mathbb{R}^{n \times n}$ and A is invertible then $AB = 0$ implies $B = 0$.

Example:

$$B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}, A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \text{ are inverses}$$

$$BA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example:

The matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ is singular.

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{11} & b_{12} \end{bmatrix} = I \Rightarrow b_{11} = 1 \text{ and } b_{11} = 0. \\ \text{(Contradiction)}$$

Theorem - Inverses are unique

proof -

Suppose $AB=I$, $BA=I$, $AC=I$, $CA=I$.

$$\Rightarrow AB=I$$

$$\Rightarrow CAB=CI$$

$$\Rightarrow I \cdot B = C \cdot I$$

$$\Rightarrow B=C.$$

Definition - If A is invertible we define the inverse of A by A^{-1} .

Example:

Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} b_{11} + 2b_{21} + 3b_{31} & b_{12} + 2b_{22} + 3b_{32} & b_{13} + 2b_{23} + 3b_{33} \\ 2b_{11} + 5b_{21} + 3b_{31} & 2b_{12} + 5b_{22} + 3b_{32} & 2b_{13} + 5b_{23} + 3b_{33} \\ b_{11} + 8b_{31} & b_{12} + 8b_{32} & b_{13} + 8b_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{array}{l|l|l} b_{11} + 2b_{12} + 3b_{13} = 1 & b_{12} + 2b_{22} + 3b_{23} = 0 & b_{13} + 2b_{23} + 3b_{33} = 0 \\ 2b_{11} + 5b_{21} + 3b_{31} = 0 & 2b_{22} + 5b_{22} + 3b_{23} = 1 & 2b_{13} + 5b_{23} + 3b_{33} = 0 \\ b_{11} + 8b_{31} = 0 & b_{12} + 8b_{23} = 0 & b_{13} + 8b_{33} = 1. \end{array}$$

↓

Augmented Matrix.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \\ -2R_1 \\ -R_1 \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array} \right] \begin{array}{l} -2R_2 \\ \\ +2R_2 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -1 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array} \right] \begin{array}{l} \\ \\ \times (-1) \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -1 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array} \right] \begin{array}{l} -9R_3 \\ +3R_3 \\ \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array} \right]$$

$$b_{11} = -40, \quad b_{12} = 16, \quad b_{13} = 9$$

$$b_{21} = 13, \quad b_{22} = -5, \quad b_{23} = -3$$

$$b_{31} = 5, \quad b_{32} = -2, \quad b_{33} = -1.$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

Example:

Solve the following system of equations

$$2x + 6y + 6z = a$$

$$2x + 7y + 6z = b \Rightarrow A\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow \vec{x} = A^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$2x + 7y + 7z = c$$

Augmented matrix:

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \div 2 \\ \\ \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & \frac{1}{2} & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \\ -2R_1 \\ -2R_1 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array} \right] \begin{array}{l} -3R_2 \\ \\ -R_2 \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & \frac{3}{2} & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right] \begin{array}{l} -3R_3 \\ \\ \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right] \Rightarrow A^{-1} = \begin{bmatrix} \frac{3}{2} & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{3}{2}a \\ -a+b \\ -b+c \end{bmatrix}$$

$$\boxed{\begin{array}{l} x = \frac{3}{2}a \\ y = b - a \\ z = c - b \end{array}}$$

Example:

If $A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$, find A^{-1} .

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \\ -2R_1 \\ +R_1 \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 0 \end{array} \right] \begin{array}{l} \\ \\ +R_2 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{array} \right] \Rightarrow \begin{array}{l} -1=0? \\ 1=0? \end{array} \Rightarrow \text{No inverse!}$$

Theorem - The following statements are equivalent.

- A is invertible
- $A\vec{x} = \vec{0}$ has only the trivial solution.
- Reduced row echelon form of A is I .
- $A\vec{x} = \vec{b}$ has only one solution for all \vec{b} .

proof:

If A is invertible then it can be row reduced to the identity.

Theorem - If A, B are invertible then

$$(AB)^{-1} = B^{-1}A^{-1}$$

proof:

$$(A \cdot B) \cdot (B^{-1} \cdot A^{-1}) = AB \cdot B^{-1}A^{-1} = A \cdot I \cdot A^{-1} = A \cdot A^{-1} = I.$$