

Section 2.2: Subspaces and Spanning Sets.

Definition - A subset W of V is a subspace of V if W is also a vector space under scalar multiplication and addition in V .

Example:

Let W be vectors of the form

$$\begin{bmatrix} x \\ x \end{bmatrix},$$

$$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : y = x \right\}, \quad V = \mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \right\}.$$

W is a subspace.

proof:

1. Let $\vec{v}_1, \vec{v}_2 \in W$. Therefore, there exists $v_1, v_2 \in \mathbb{R}$ such that

$$\vec{v}_1 = \begin{bmatrix} v_1 \\ v_1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} v_2 \\ v_2 \end{bmatrix}$$

$$\Rightarrow \vec{v}_1 + \vec{v}_2 = \begin{bmatrix} v_1 + v_2 \\ v_1 + v_2 \end{bmatrix} \in W$$

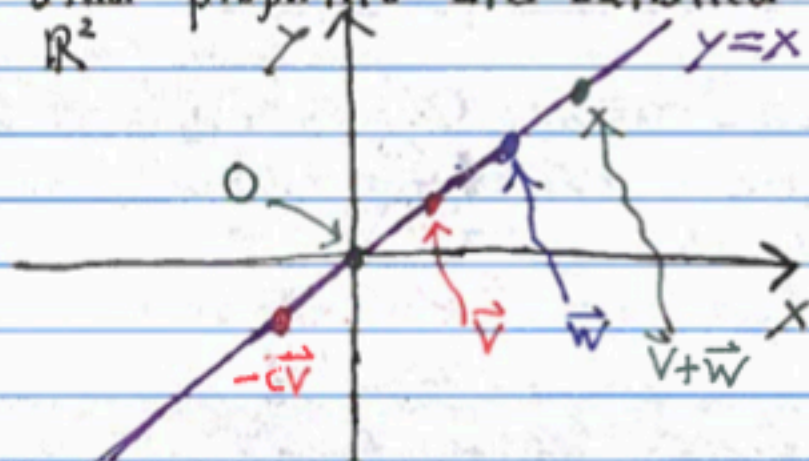
} closure.

$$2. c\vec{v}_1 = \begin{bmatrix} c\vec{v}_1 \\ c\vec{v}_1 \end{bmatrix} \in W$$

$$3. 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in W$$

$$4. -\vec{v}_1 = \begin{bmatrix} -v_1 \\ -v_1 \end{bmatrix} \in W.$$

All other properties are satisfied by default.



Theorem - Let W be a subset of a vector space V .

W is a subspace of V if and only if:

(a) $\vec{u}, \vec{v} \in W \Rightarrow \vec{u} + \vec{v} \in W$

(b) $\vec{u} \in W \Rightarrow c\vec{u} \in W$

Examples:

1. $W = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ is not a subspace of \mathbb{R}^2 .

proof:

$$(1, 1) \in W \Rightarrow -1(1, 1) = (-1, -1) \notin W.$$

2. $W = \{n \times n \text{ symmetric matrices}\}$ is a subspace of $\mathbb{R}^{n \times n}$.

proof:

- Sum of two symmetric matrices is symmetric.

- Scalar times symmetric matrix is symmetric.

Theorem - If $A\vec{x} = 0$, the set of homogeneous solutions is a subspace of \mathbb{R}^n . (Nullspace, kernel)

proof:

a.) Suppose \vec{x}_1, \vec{x}_2 satisfy $A\vec{x}_1 = 0, A\vec{x}_2 = 0$. Therefore,

$$\begin{aligned} A(\vec{x}_1 + \vec{x}_2) &= A\vec{x}_1 + A\vec{x}_2 \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

b.) Suppose \vec{x}_1 satisfies $A\vec{x}_1 = 0$. Therefore,

$$\begin{aligned} A(c\vec{x}_1) &= cA\vec{x}_1 \\ &= c \cdot 0 \\ &= 0. \end{aligned}$$

Example:

Describe the nullspace of the following matrices

$$\text{a.) } A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{bmatrix} \begin{array}{l} -2R1 \\ -3R1 \end{array} \Rightarrow \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$z = \text{anything}$$

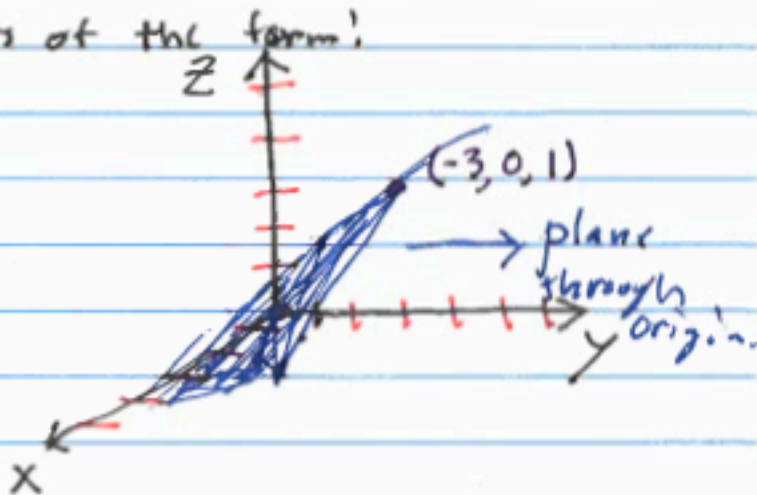
$$y = \text{anything}$$

$$x - 2y + 3z = 0$$

$$\Rightarrow x = 2y - 3z$$

$NS(A) =$ set of vectors of the form:

$$\begin{bmatrix} 2y - 3z \\ y \\ z \end{bmatrix}$$

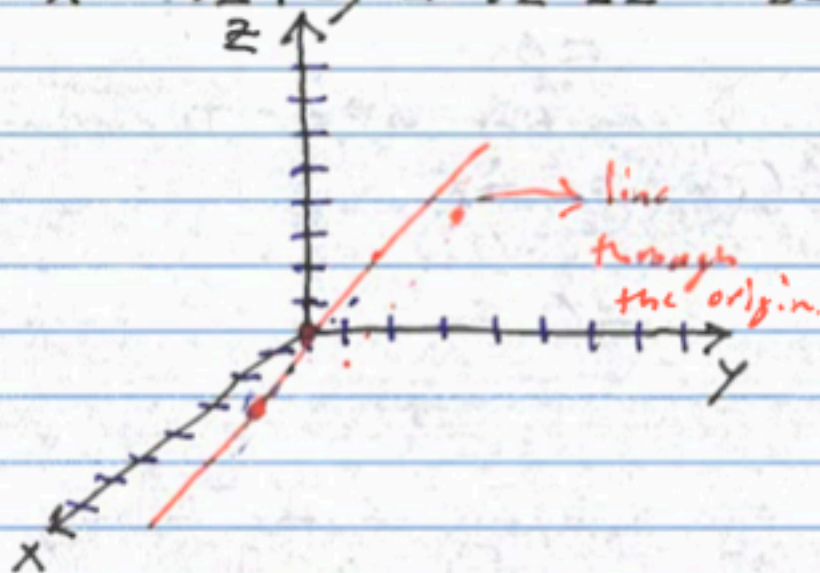


$$b.) \begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ -2 & 4 & -6 \end{bmatrix} \begin{array}{l} +3R_1 \\ +2R_1 \end{array} \Rightarrow \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow z = \text{any real}$

$$y = -z$$

$$x = -3z + 2z = -3z - 2z = -5z.$$



$$c.) \begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ 4 & 1 & 2 \end{bmatrix} \begin{array}{l} +3R_1 \\ -4R_1 \end{array} \Rightarrow \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 1 \\ 0 & 9 & -10 \end{bmatrix} \begin{array}{l} \\ -9R_2 \end{array} \Rightarrow \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -19 \end{bmatrix}$$

$$NA(A) = \{0\}.$$

$$d.) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow NA(A) = \mathbb{R}^3.$$

Definition - A vector \vec{w} is a linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ if it can be written in the form

$$\vec{w} = k_1 \vec{v}_1 + \dots + k_n \vec{v}_n,$$

where $k_1, \dots, k_n \in \mathbb{R}$.

Example:

1. Express $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + k_3 \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow k_1 = 1, k_2 = 2, k_3 = 0$$

$$\Rightarrow \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

2. Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$.

a.) Is $\vec{w} = \begin{bmatrix} 9 \\ 2 \end{bmatrix}$ a linear combination of \vec{u} and \vec{v} ?

b.) Is $\vec{w} = \begin{bmatrix} 9 \\ 7 \end{bmatrix}$ is not a linear combination of \vec{u} and \vec{v} .

$$\begin{bmatrix} 9 \\ 2 \\ 7 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + k_2 \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{array} \right] \xrightarrow[-R1]{+R1} \left[\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 8 & 16 \end{array} \right] \xrightarrow{+R2} \left[\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right]$$

$$\Rightarrow k_2 = 2$$

$$k_1 = -3.$$

$$\Rightarrow \vec{w} = -3\vec{u} + 2\vec{v}.$$

$$3. \left[\begin{array}{cc|c} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 8 \end{array} \right] \xrightarrow[-R1]{+R1} \left[\begin{array}{cc|c} 1 & 6 & 4 \\ 0 & -8 & -9 \\ 0 & 8 & 12 \end{array} \right]$$

No solution \Rightarrow Not a linear combination.

Theorem - If V is a vector space and $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are vectors in V , then the set of all linear combinations called the subspace of V spanned by $\vec{v}_1, \dots, \vec{v}_n$ denoted by $\text{span}\{\vec{v}_1, \dots, \vec{v}_n\}$ is a subspace of V .

proof:

If $\vec{v}, \vec{w} \in V$ then there exists $c_1, \dots, c_n, d_1, \dots, d_n$ so that

$$\vec{v} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n$$

$$\vec{w} = d_1 \vec{v}_1 + \dots + d_n \vec{v}_n$$

$$\Rightarrow \vec{v} + \vec{w} = (c_1 + d_1) \vec{v}_1 + \dots + (c_n + d_n) \vec{v}_n \in \text{span}\{\vec{v}_1, \dots, \vec{v}_n\}$$

$$k\vec{v} = k(c_1 \vec{v}_1 + \dots + c_n \vec{v}_n)$$

$$= kc_1 \vec{v}_1 + \dots + kc_n \vec{v}_n \in \text{span}\{\vec{v}_1, \dots, \vec{v}_n\}$$

Example:

1. Is $2x^2 + x + 1$ in $\text{span}\{x^2 + x, x^2 - 1, x + 1\}$?

$$a(x^2 + x) + b(x^2 - 1) + c(x + 1) = 2x^2 + x + 1$$

$$(a+b)x^2 + (a+c)x + (c-b) = 2x^2 + x + 1$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 & | & 2 \\ 1 & 0 & 1 & | & 1 \\ 0 & -1 & 1 & | & 1 \end{bmatrix} \xrightarrow{-R1} \begin{bmatrix} 1 & 1 & 0 & | & 2 \\ 0 & -1 & 1 & | & -1 \\ 0 & -1 & 1 & | & 1 \end{bmatrix} \xrightarrow{-R2} \begin{bmatrix} 1 & 1 & 0 & | & 2 \\ 0 & -1 & 1 & | & -1 \\ 0 & 0 & 0 & | & -2 \end{bmatrix}$$

There is no solution and thus $2x^2 + x + 1$ is not in $\text{span}\{x^2 + x, x^2 - 1, x + 1\}$.

2. Do $\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ span \mathbb{R}^2 . Let $\vec{v} = (v_1, v_2)$ if $\vec{v} \in \text{span}\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \end{bmatrix}\}$ then

$$c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -4 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 2 & v_1 \\ -2 & -4 & v_2 \end{array} \right] + 2R_1$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 2 & v_1 \\ 0 & 0 & v_2 + 2v_1 \end{array} \right]$$

This does not have a solution for all \vec{v}_1, \vec{v}_2 .
 $\Rightarrow \{\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \end{bmatrix}\}$ does not span \mathbb{R}^2 .