Section 2.3: Linear Independence and Bases

Definition- The vectors $\vec{v}_1, \dots, \vec{v}_n$ are linearly dependent if one of the vectors can be written as a linear combination of the others i.e,

$$\vec{v}_1 = c_2\vec{v}_2 + c_3\vec{v}_3 + \dots + c_n\vec{v}_n$$ → Useful for understanding

i.e. $\vec{v}_1$ depends on the others.

Equivalently, $\vec{v}_1, \dots, \vec{v}_n$ are linearly dependent if there are scalars $c_1, \dots, c_n$ not all zero so that

$$c_1\vec{v}_1 + \dots + c_n\vec{v}_n = 0.$$ → Useful for problems.

If $\vec{v}_1, \dots, \vec{v}_n$ are not linearly dependent, they are linearly independent.

Examples

1. $\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 5 \\ -1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 7 \\ -1 \\ 5 \\ 8 \end{bmatrix}$, are linearly dependent.

$$3\vec{v}_1 + \vec{v}_2 - \vec{v}_3 = \begin{bmatrix} 6 \\ -3 \\ 0 \\ 9 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 5 \\ -1 \end{bmatrix} - \begin{bmatrix} 7 \\ -1 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \vec{v}_3 = 3\vec{v}_1 + \vec{v}_2$ ($\vec{v}_3$ depends on $\vec{v}_1$ and $\vec{v}_2$).

* purple depends on red and blue.

2. $p_1 = 1 - x$, $p_2 = 5 + 3x - 2x^2$, $p_3 = 1 + 3x - x^2$ are linearly dependent

$$3p_1 - p_2 + 2p_3 = 0$$

$\rightarrow p_2 = 3p_1 + 2p_3$

3. The vectors

$$\vec{i} = \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{j} = \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{k} = \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are linearly independent.

4. Are the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

linearly dependent or linearly independent.

$$c_1 \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} c_1 + 5c_2 + 3c_3 &= 0 \\ -2c_1 + 6c_2 + 2c_3 &= 0 \\ 3c_1 - c_2 + c_3 &= 0 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{bmatrix} \begin{array}{l} \\ +2R1 \\ -3R1 \end{array} \Rightarrow \begin{bmatrix} 1 & 5 & 3 \\ 0 & 16 & 8 \\ 0 & -16 & -8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 5 & 3 \\ 0 & 16 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

There are nontrivial solutions

$\Rightarrow$ Linearly dependent.

$$\begin{aligned} \det(A) &= 1 \cdot (6+2) - 5(-2-6) + 3(2-18) \\ &= 8 + 40 - 48 \\ &= 0. \end{aligned}$$

Lets try to solve for one of the vectors

c_3 = anything

$16c_2 + 8c_3 = 0$

$c_2 = -\frac{1}{2}c_3$

finally

$$c_1 + 5c_2 + 3c_3 = 0$$

$$c_1 - \frac{5}{2}c_3 + 3c_3 = 0$$

$$c_1 = -\tfrac{1}{2}c_3$$

$$\Rightarrow -\frac{1}{2}c_3\vec{v}_1 - \frac{1}{2}c_3\vec{v}_2 + c_3\vec{v}_3 = 0$$

$$\Rightarrow \vec{v}_3 = \frac{1}{2}\vec{v}_1 + \frac{1}{2}\vec{v}_2$$

Definition - We say that the vectors $\vec{v}_1, \dots, \vec{v}_n$ are a basis for a vector space V are a basis for V if both of the following are satisfied:

1. $\vec{v}_1, \dots, \vec{v}_n$ are linearly independent.
2. $\vec{v}_1, \dots, \vec{v}_n$ span V.

Example:

1. Show that

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

forms a basis for $\mathbb{R}^3$.

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} -R1 \Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} /-2 \Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Yes, linearly ind, also a basis.

2. $p_1 = x^2 + x - 3$, $p_2 = x - 5$, $p_3 = 3$

$$c_1(x^2 + x - 3) + c_2(x-5) + 3c_3 = 0$$

$$\begin{aligned} c_1 &= 0 \\ c_1 + c_2 &= 0 \\ -3c_1 - 5c_2 + 3c_3 &= 0 \end{aligned} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -3 & -5 & 3 \end{bmatrix} \begin{array}{l} \\ \Rightarrow -R1 \\ \Rightarrow +3R1 \end{array} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 3 \end{bmatrix}$$

$\Rightarrow c_1 = 0$

$c_2 = 0 \Rightarrow$ linear independence, basis.

$c_3 = 0$

3. $p_1 = x^2 + x - 1$, $p_2 = x^2 - x + 1$

Suppose

$$ax^2 + bx + c = c_1 p_1 + c_2 p_2$$
$$ax^2 + bx + c = (c_1 + c_2)x^2 + (c_1 - c_2)x + (c_2 - c_1)$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 1 & a \\ 1 & -1 & b \\ -1 & 1 & c \end{array}\right] \begin{array}{l} \\ -R1 \\ +R1 \end{array} \Rightarrow \begin{bmatrix} 1 & 1 & a \\ 0 & -2 & a-b \\ 0 & 2 & a+c \end{bmatrix}$$

$$\Rightarrow c_2 = \frac{b-a}{2}, \quad c_2 = \frac{a+c}{2}$$

Not consistent $\Rightarrow$ Not a basis.

4. $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$$\begin{array}{l} c_1 - c_2 + 2c_3 = 0 \\ c_1 + c_2 + 3c_3 = 0 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 1 & 1 & 3 & 0 \end{array}\right] \begin{array}{l} \\ -R1 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & -2 & 1 & 0 \end{array}\right]$$

$c_3 = 2c_2 \Rightarrow$ linear dependent $\Rightarrow$ not a basis.