MST 205
Fall 2021
Exam #1
09/16/21

Name (Print): Key

The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Short answer questions:** Questions labeled as **"Short Answer"** can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit.** An incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Problem	Points	Score
1	15	
2	15	
3	15	
4	15	
5	10	
6	15	
7	10	
8	5	
Total:	100	

Do not write in the table to the right.

1. (15 points) (**Short Answer**) Determine if the following statement is correct (**C**) or incorrect (**I**). Just circle **C** or **I**. No need to show any work. In order for a statement to be correct it must be true in all cases.

   C **I** If A and B are $n \times n$ matrices then $(AB)^2 = A^2B^2$.

   C **I** If A is an $n \times n$ matrix satisfying $\det(A) = 0$ then $A\vec{x} = 0$ has infinitely many solutions.

   C **I** If A is an $n \times n$ matrix and k is a scalar then $\det(kA) = k\det(A)$.

   C **I** If A is an $n \times n$ matrix then $\det(A^2) = \det(A)^2$.

   C **I** If A is an invertible $n \times n$ matrix then the only solution of the system $A\vec{x} = \vec{x}$ is $\vec{x} = 0$.

2. (15 points) Solve the following system of linear equations:

$$x + y + 2z = 9,$$
$$2x + 4y - 4z = 2,$$
$$3x + 6y - 6z = 0.$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -4 & 2 \\ 3 & 6 & -6 & 0 \end{array}\right] \begin{array}{l} \\ -2R1 \\ -3R1 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -8 & -16 \\ 0 & 3 & -12 & -27 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -4 & -8 \\ 0 & 1 & -4 & -9 \end{array}\right]$$

Row 2 and 3 are inconsistent $\Rightarrow$ No solution.

3. (15 points) Let

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 5 \\ -1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}.$$

Compute the following or state that the computation is impossible.

(a) (5 points) $-3(B - 2C)$

$$-3(B-2C) = -3\begin{bmatrix} 2 & -11 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} -6 & 33 \\ -6 & -6 \end{bmatrix}$$

(b) (5 points) AB

$$\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 12 & -3 \\ -4 & 5 \\ 4 & 1 \end{bmatrix}$$

(c) (5 points) DA^T.

Not possible.

4. (15 points) Find the inverse of the following matrix or state why it is not invertible.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]\begin{array}{l} \\ -R1 \\ \\ \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]\begin{array}{l} \\ \\ -R2 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -2 & 1 & -1 & 1 \end{array}\right]\begin{array}{l} \\ \\ /2 \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right]\begin{array}{l} -R3 \\ +R3 \\ \\ \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

5. (10 points) Suppose A is an $n \times n$ matrix satisfying $A^2 = 0$ and let $B = I - A$.

(a) (5 points) Show that $A^3 = 0$.

$A^3 = A^2 \cdot A = 0 \cdot A = 0$

(b) (5 points) Show that

$$B^{-1} = I + A.$$

$(I-A)(I+A) = I - AI + IA + A^2$

$= I - A + A + 0$

$= I$

Therefore, $(I-A)^{-1} = I + A$.

6. (15 points) Calculate the determinants of the following matrices where $k \in \mathbb{R}$ is a constant.

(a) (5 points) $A = \begin{bmatrix} -3 & 0 & 7 \\ 2 & 5 & 0 \\ -1 & 0 & 5 \end{bmatrix}$

$$\det(A) = 5\det\left(\begin{bmatrix} -3 & 7 \\ -1 & 5 \end{bmatrix}\right) = 5(-15+7) = -40$$

(b) (5 points) $A = \begin{bmatrix} 1 & k & k^2 \\ 1 & k & k^2 \\ 1 & k & k^2 \end{bmatrix}$

$$\det(A) = \det\begin{pmatrix} 1 & k & k^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

(c) (5 points) $A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & -3 \end{bmatrix}$

$$\det(A) = 2 \cdot 1 \cdot (-3) = -6.$$

7. (10 points) For what values of k does the following system of equations have no solutions, a single solution, or infinitely many solutions?

$$x - y = 3,$$
$$-k(k-2)y = k.$$

- Infinite solutions if $K=0$
- No solution if $K=2$
- One solution if $K\neq 2$ and $K\neq 0$.

8. (5 points) Let A and B be $n \times n$ invertible matrices. Show that $\det(B) = \det(A^{-1}BA)$.

$$\det(A^{-1}BA) = \det(A^{-1})\det(B)\det(A)$$
$$= \frac{\det(A)}{\det(A)} \cdot \det(B)$$
$$= \det(B).$$