Homework #6

#7.1

$$\dot{S} = \Lambda - \beta S I - \nu S + \phi I$$
$$\dot{I} = \beta S I - (\phi + \nu + \alpha) I$$

(a) We can nondimensionalize this system by setting

$$x = \frac{\nu}{\Lambda} S, \quad y = \frac{\nu}{\Lambda} I, \quad \tau = (\phi + \nu + \alpha) t$$

$$\Rightarrow \frac{\Lambda}{\nu}(\phi+\nu+\alpha)\frac{dx}{d\tau} = \Lambda \quad \beta \frac{\Lambda^2}{\nu^2} xy - \Lambda x + \frac{\phi \Lambda}{\nu} y$$

$$\frac{\Lambda}{\nu}(\phi+\nu+\alpha)\frac{dy}{d\tau} = \beta \frac{\Lambda^2}{\nu^2} xy - (\phi+\nu+\alpha)\frac{\Lambda}{\nu} y$$

$$\Rightarrow \frac{dx}{d\tau} = \frac{\nu}{\phi+\nu+\alpha} - \frac{\beta \Lambda}{\nu(\phi+\nu+\alpha)} xy - \frac{\nu}{(\phi+\nu+\alpha)} x + \frac{\phi}{\phi+\nu+\alpha} y$$

$$\frac{dy}{d\tau} = \frac{\beta \Lambda}{\nu(\phi+\nu+\alpha)} xy - y$$

$$\Rightarrow \frac{dx}{d\tau} = A - R_0 xy - Ax + By,$$

$$\frac{dy}{d\tau} = R_0 xy - y,$$

where

$$R_0 = \frac{\beta \Lambda}{\nu(\phi+\nu+\alpha)}, \quad A = \frac{\nu}{(\phi+\nu+\alpha)}, \quad B = \frac{\phi}{\phi+\nu+\alpha}.$$

The disease free equilibrium is therefore given by

$$(x_1^*, y_1^*) = (1, 0)$$

Now, consider the function

$$V(x,y) = \frac{1}{2}[(x-1)+y]^2 + Ky,$$

where K is a constant to be chosen. It follows that

$$\nabla V = (((x-1)+y), (x-1)+y+K)$$

$$\Rightarrow \nabla V \cdot (\dot{x}, \dot{y}) = (x-1+y)\cdot(A - R_0 xy - Ax + By)$$
$$+ (x-1+y+K)(R_0 xy - y)$$
$$= A(x-1+y) - Ax(x-1+y) + By(x-1+y)$$
$$+ K R_0 xy - y(x-1+y+K)$$
$$= Ax - A + Ay - Ax^2 + Ax - Axy + Bxy - By + By^2$$
$$+ K R_0 xy - yx + y - y^2 - Ky.$$

To eliminate the cross term we choose

$$B - A + K R_0 - 1 = 0$$
$$\Rightarrow K = \frac{A-B+1}{R_0}.$$

Therefore,

$$\frac{dV}{dt} = -A(x^2 - 2x + 1) + Ay + By^2 - By + y - y^2 - \frac{(A-B+1)}{R_0}y$$
$$= -A(x-1)^2 + (B-1)y^2 - \left(B - A - 1 - \left(\frac{B-A-1}{R_0}\right)\right)y$$

Now

$$0 < B = \frac{\phi}{\phi + \nu + \alpha} < 1, \quad 0 < A = \frac{\nu}{\phi + \nu + \alpha} < 1$$

and if we assume $R_0 < 1$ it follows that

$$-1 < B - A < 1$$
$$\Rightarrow -2 < B - A + 1 < 0$$

Therefore, since $B - A + 1 < 0$ it follows that

$$\frac{B-A+1}{R_0} < B - A + 1$$

Therefore,

$$\frac{dV}{dt} \leq -A(x-1)^2 - (1-B)y^2 \leq 0.$$

(b). The endemic equilibrium is given by:

$$x^* = 1/R_0, \quad y^* = \frac{A(R_0-1)}{(1-B)R_0}$$

Define the Lyapunov function by:

$$V = \frac{1}{2}\left[(x-x^*)+(y-y^*)\right]^2 + \frac{A-B+1}{R_0}\left(y-y^*-y^*\ln\left(\frac{y}{y^*}\right)\right).$$

Therefore,

$$\nabla V = \left(x-x^*+y-y^*,\ x-x^*+y-y^* + \frac{A-B+1}{R_0}\left(1-\frac{y^*}{y}\right)\right).$$

Consequently,

$$\frac{dV}{dt} = \nabla V\cdot(\dot{x},\dot{y}) = (x-x^*+y-y^*)(A-R_0xy-Ax+By) + \left(x-x^*+y-y^*+\frac{A-B+1}{R_0}\left(1-\frac{y^*}{y}\right)\right)(R_0xy-y)$$

Simplifying we have that

$$\frac{dV}{dt} = -Ax^2 - \frac{A^2(R_0-1)^2}{(1-B)R_0} + \frac{2AX}{R_0} - \frac{A}{R_0^2} + 2Ay - \frac{2Ay}{R_0} + (1-B)y^2$$

Again, since $A<1, B<1$ if we assume $R_0 \geq 1$ then

$$-Ax^2 < -\frac{A}{R_0}x^2, \quad -(1-B)y^2 < -2y^2, \quad 2Ay - \frac{2Ay}{R_0} < 0$$

and thus

$$\frac{dV}{dt} < -\frac{A}{R_0}(x-1)^2 - \frac{A^2(R_0-1)^2}{(1-B)R_0} - (1-B)y^2 < 0.$$

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