MST 383/683
Homework #2

Due Date: September 09, 2021

1. pg. 31, #2.8
2. The curves $x(t)$ illustrated below correspond to solution curves for the differential equation $\dot{x} = f(x)$.

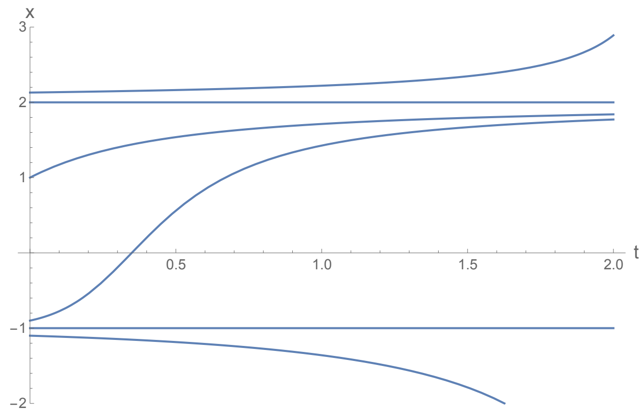

Figure 1:

   (a) Sketch a one dimensional phase portrait that is consistent with Fig. 1.
   (b) Sketch a graph of $f(x)$ that is consistent with Fig. 1.
   (c) Give a formula for $f(x)$ that is consistent with Fig. 1.

3. Consider the following SIS model with constant population size N:

$$\dot{S} = -\frac{\beta I^p S}{1 + (\sigma I)^q} + \alpha I,$$
$$\dot{I} = \frac{\beta I^p S}{1 + (\sigma I)^q} - \alpha I.$$

   (a) Determine the dimensions of the positive constants σ, β and α.
   (b) Reduce the SIS model to a single differential equation.
   (c) Introducing an appropriate dimensionless population x and dimensionless time τ, convert the single differential equation into dimensionless form.
   (d) For the case $p < 1$, $q = p - 1$, determine the threshold condition for the existence of endemic equilibria.
   (e) For the case $p > 1$, $p = q$, determine the threshold condition for the existence of endemic equilibria.
   (f) Why does this model not make sense if $p > 1$ and $p > q$?.

4. In cases of constant recruitment, the limiting system as $t \to \infty$ and the original system usually have the same qualitative dynamics. Consider the following SIV model with constant recruitment and vaccination:

$$\dot{S} = \Lambda - \beta S \frac{I}{N} - \mu S$$
$$\dot{I} = \beta S \frac{I}{N} - (\mu + \gamma) I$$
$$\dot{V} = \gamma I - \mu V,$$

where again $N(t) = S(t) + I(t) + V(t)$ and all of the parameters are positive constants.

(a) What are the units of Λ, β, μ and γ?

(b) Interpret in practical terms what the constants Λ, β, μ, and γ represent physically.

(c) Find an equation for $\dot{N}(t)$ and solve this equation for $N(t)$.

(d) Show that $\lim_{t\to\infty} N(t) = \Lambda/\mu$.

(e) Explain why part (d) allows us to consider the following equivalent system:

$$\begin{aligned}\dot{S} &= \mu N - \beta S I - \mu S\\ \dot{I} &= \beta S I - (\mu + \gamma) I\\ \dot{V} &= \gamma I - \mu V.\end{aligned}$$

(f) Reduce the second system of two equations to a system of two differential equations for the infected and susceptible populations that is independent of V.

5. A disease is introduced by two visitors into a town with 1200 inhabitants. An average infective is in contact with 0.4 inhabitants per day. The average duration of the infective period is 6 days, and recovered infectives are immune against reinfection. Assuming an SIR model, estimate how many inhabitants would have to be immunized to avoid an epidemic.