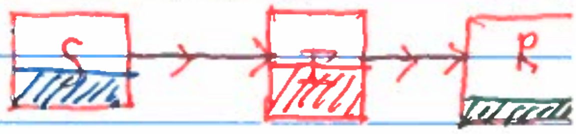
Section 2.1: SIR Model

How do we model the spread of a disease?
First ODE Model:

$S \sim$ proportion of susceptible population.
$I \sim$ proportion of infected population.
$R \sim$ proportion of removed
$N \sim$ total population.

Assumptions:



1. N is constant in time
2. $\frac{dS}{dt} = \dot{S} = -f(S,I)$ ← Incidence.
3. $\frac{dI}{dt} = \dot{I} = f(S,I) - g(I)$ ← Recovery
4. $\frac{dR}{dt} = \dot{R} = g(I)$

$\Rightarrow \dot{N} = \dot{S} + \dot{I} + \dot{R} = -f + f - g + g = 0$ (pop. conser

5. $f(0,I) = 0$, $f(S,0) = 0$, $f \geq 0$

   *Simplest function:

   $$f(S,I) = \beta S I$$

Interpretation:

$$\frac{S(t+\Delta t) - S(t)}{\Delta t} \approx -\beta S(t) I(t)$$

$$\Rightarrow S(t+\Delta t) \approx S(t) - \underbrace{\beta \Delta t\, I(t)}_{\text{proportion of } S \text{ infected}} S(t)$$

"probability of infection"

βI ~ force of infection per unit time.

6. $g(0)=0$, $g \geq 0$.
* simplest function
$$g(I) = \alpha I$$

Interpretation:
$$R(t+\Delta t) \approx R(t) + \underbrace{\alpha \Delta t}\, I(t).$$

proportion of I that recover.
"probability of recovery"

Full Model:

$$(*) \quad \begin{aligned} \dot{S} &= -\beta I S \\ \dot{I} &= \beta I S - \alpha I \\ \dot{R} &= \alpha I \end{aligned} \quad , \quad \left.\begin{aligned} S(0) &= S_0 \\ I(0) &= I_0 \\ R(0) &= R_0 \end{aligned}\right\} \text{Initial Conditions} \Rightarrow N = S_0 + I_0 + R_0.$$

How do we analyze??

Theorem - Solutions to $(*)$ exist and are unique.

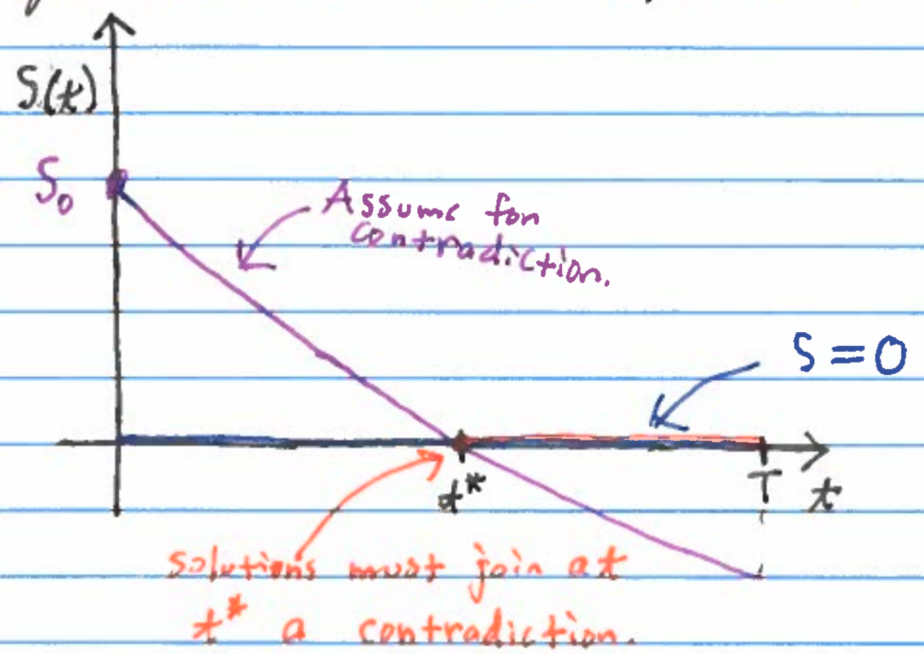
Corollary - If $S_0, I_0, R_0 > 0$ then for all time $S(t), I(t), R(t) >$

proof:
$S(t) = 0$ is a solution for all t. For $S_0 > 0$ let $S(t)$ be a solution satisfying $S(0) = S_0$. For contradiction suppose there exists some $T > 0$ such that $S(T) \leq 0$.

By the intermediate value theorem there exists $t^* \in [0,T]$ such that $S(t^*)=0$. Consequently,

$$\dot{S}(t^*) = 0$$
$$\Rightarrow S=0 \text{ for all } t \geq t^*.$$

This violates existence and uniqueness. Similar arguments prove that $I(t), R(t) > 0$.



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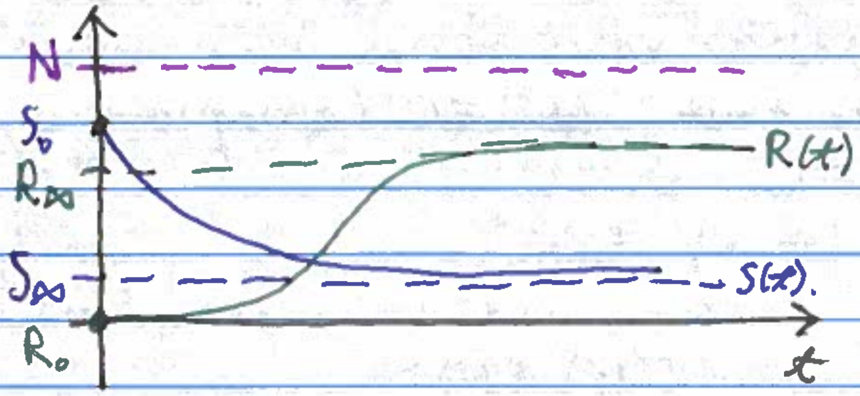
Corollary: For all $S_0, I_0, R_0 > 0$ there exists $S_\infty, R_\infty \geq 0$ such that

$$\lim_{t\to\infty} S(t) = S_\infty \quad \text{and} \quad \lim_{t\to\infty} R(t) = R_\infty.$$

~~proof:~~

1. Since $S(t) \geq 0$ and $\dot{S}(t) \leq 0$ it follows that S is monotone decreasing and bounded below. Therefore, there exists $S_\infty \geq 0$ such that $\lim_{t\to\infty} S(t) = S_\infty$.

2. Since $R(t) \geq 0$ and $R(t) \leq N$ it follows that R is monotone increasing and bounded above. Therefore, there exists R_∞ such that $\lim_{t\to\infty} R(t) = R_\infty$.



What about $I(t)$?

$$\dot{I}(0) = (\beta S(0) - \alpha) I(0).$$

Therefore, if $\beta S(0) - \alpha > 0$ $\dot{I}(t) > 0$

$$\Rightarrow \boxed{\frac{\beta S(0)}{\alpha} > 1}$$ Condition for an epidemic

Usually, $S(0) \approx N$ and thus

$$\boxed{R_0 = \frac{\beta N}{\alpha} > 1}$$

Basic Reproduction Number.

<u>Theorem</u>: $\lim_{t\to\infty} I(t) = I_\infty = 0.$

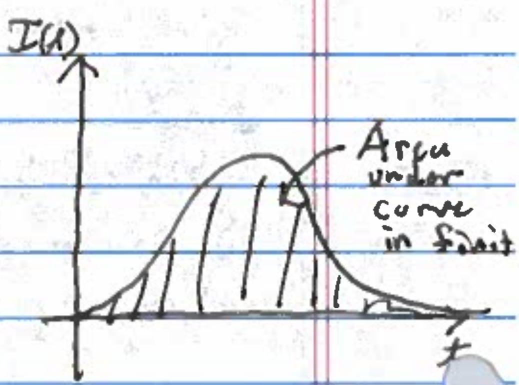
proof:

$$\int_0^\infty \dot{S}(t)\,dt = -\beta \int_0^\infty S(t) I(t)\,dt$$

$$\Rightarrow S_\infty - S_0 = -\beta \int_0^\infty S(t) I(t)\,dt$$

$$\Rightarrow S_0 - S_\infty = \beta \int_0^\infty S(t) I(t)\,dt$$

$$\Rightarrow S_0 - S_\infty \geq \beta S_\infty \int_0^\infty I(t)\,dt \geq 0.$$



Therefore, $\lim_{t\to\infty} I(t) = 0.$ ∎

What is the maximum number of infected?

$$\dot{I} = \beta I S - \alpha I$$

$$\dot{I} = 0 \Rightarrow I(\beta S - \alpha) = 0$$

$$\Rightarrow S = \alpha/\beta.$$

When I is maximized $S = \alpha/\beta$. Need to connect S and I somehow:

$$\frac{\dot{S}}{\dot{I}} = \frac{dS}{dI} = \frac{-\beta I S}{\beta I S - \alpha I} = \frac{-S}{\beta S - \alpha}$$

$$\Rightarrow \int_{S_0}^{S} \frac{\alpha - \beta S}{\beta S} dS = \int_{I_0}^{I} dI$$

$$\Rightarrow \alpha \ln\left(\frac{S}{S_0}\right) - \beta(S - S_0) = I - I_0.$$

$$\Rightarrow \boxed{I = I_0 + \frac{\alpha}{\beta} \ln\left(\frac{S}{S_0}\right) - (S - S_0)}$$

Therefore

$$\boxed{I_{max} = I_0 + \frac{\alpha}{\beta} \ln\left(\frac{\alpha}{\beta S_0}\right) - (\alpha/\beta - S_0).}$$

Now, if $I_0 \approx 0$ and $S_0 \approx N$ we obtain:

$$I_{max} \approx \frac{\alpha}{\beta} \ln\left(\frac{\alpha}{\beta N}\right) + N - \frac{\alpha}{\beta}$$

$$\rightarrow I_{max} \approx N - \frac{\alpha}{\beta}\left(1 + \ln(R_0)\right)$$

$$\boxed{I_{max} \approx N\left(1 - \frac{1}{R_0}\left(1 + \ln(R_0)\right)\right)}$$

Now, we can also solve for S_∞. At the end of the epidemic we know $I=0$ and assume $S_0 \approx N$

$$\Rightarrow 0 = I_0 + \alpha/\beta \ln(S_\infty/S_0) - (\alpha/\beta - S_0)$$

$$\Rightarrow 0 = I_0 + \frac{N}{R_0} \ln\left(\frac{S_\infty}{S_0}\right) - \frac{N}{R_0} + S_0$$

$$\Rightarrow \frac{N}{R_0} - I_0 - S_0 = \frac{N}{R_0} \ln\left(\frac{S_\infty}{S_0}\right)$$

$$\Rightarrow 1 - \frac{R_0(I_0+S_0)}{N} = \ln\left(\frac{S_\infty}{S_0}\right)$$

$$\Rightarrow \boxed{S_\infty = S_0 \exp\left(\frac{N - R_0(I_0+S_0)}{N}\right)}$$

If, $S_0 \approx N, I_0 \approx 0$

$$\Rightarrow S_\infty \approx N\exp(1-R_0)$$

$$\Rightarrow R_\infty \approx N(1 - \exp(1-R_0))$$

$$\Rightarrow \boxed{R_0 = 1 - \ln\left(\frac{S_\infty}{N}\right)}$$

If $R_0 = 2$ (Covid): $R_\infty \approx N(.63) \rightarrow 63\%$ of population

If $R_0 = 1.5$ (H1N1): $R_\infty \approx N(.4) \rightarrow 40\%$ of population

If $R_0 = 17$ (Measels): $R_\infty \approx N(1) \rightarrow 100\%$ of population.

If $R_0 = 12$ (Chicken pox): $R_\infty = N(1) \rightarrow 100\%$ of population

If $R_0 = 5$ (Polio): $R_\infty = N(.98) = 98\%$ of population

If $R_0 = 3$ (Smallpox): $R_\infty = N(.86) = 86\%$ of population