Lecture 11: Model Analysis and Sensitivity Analysis.

- Suppose, $Y_j(t_j)$ denote data points at time t_j.
- Suppose, $I(t_j)$ denote model predictions at time t_j.

$$\Rightarrow SSE = \|Y - I\|_{\ell^2}^2 = \sum_{j=1}^{n} (Y_j(t_j) - I(t_j))^2.$$

Fitting a model to data means minimizing $\|Y - I\|_{\ell^2}$ over free parameters.

Example:

For the SIR model

$$\dot{S} = -\beta S I$$
$$\dot{I} = \beta S I - \alpha I$$
$$S(0) = S_0, \; I(0) = I_0$$

there are potentially four parameters

α, β, S_0, I_0.

However, exact knowledge of the data should be used to reduce the number of paramaters.

Model Selection:

$$AIC = n \ln\left(\frac{SSE}{n}\right) + 2K$$

$n \sim$ number of data points

$K \sim$ number of parameters $+ 1$.

Find model that minimizes AIC when

$n/K \geq 40$.

For $n/K < 40$ use

$$AIC_c = n \ln\left(\frac{SSE}{n}\right) + 2K + \frac{2K(K+1)}{n-K-1}$$

$$\Delta_j = \underbrace{AIC_j - AIC_{min}}_{\text{Distance from model to minimum explanation}}$$

$0 \le \Delta_j \le 2$ support in data.

$4 \le \Delta_j \le 7$ little support in data.

$$W_i = \frac{\exp(-\Delta_i/2)}{\sum_j \exp(-\Delta_j/2)} \sim \text{weight of the } i\text{-th model.}$$

$W_i > .9 \Rightarrow$ robust inferences can be made using the best fitted model.

Example:

M_1:

$\dot{S} = -\beta SI$

$\dot{I} = +\beta SI - \alpha I$

$\dot{R} = \alpha I$

M_2:

$\dot{S} = -\beta SI$

$\dot{E} = \beta SI - \gamma E$

$\dot{I} = \gamma E - \alpha I$

$\dot{R} = \alpha I$

* Can learn exposed people.

M_3:

$\dot{S} = -\beta S(I + qA)$

$\dot{E} = \beta S(I + qA) - (\gamma + \kappa)E$

$\dot{I} = \gamma E - \alpha I$

$\dot{A} = \kappa E - \nu A$

$\dot{R} = \alpha I + \nu A$

* Can learn about asymptomatic class.

All the light we cannot see.

Sensitivity Analysis

$$\dot{y}_i = f_i(y_1, \dots, y_n, t, p)$$

$\uparrow$ parameter

How does small changes in p influence behavior of system.
We view

$$y_i : \underset{\text{time}}{\mathbb{R}} \times \underset{\text{parameter}}{\mathbb{R}} \to \mathbb{R}$$

Define,

$$z_i = \frac{\partial y_i}{\partial p}, \text{ (how does } y \text{ change with } p\text{).}$$

$$\Rightarrow \frac{dz_i}{dt} = \frac{d}{dt}\frac{\partial y_i}{\partial p} = \frac{\partial}{\partial p}\frac{dy_i}{dt} = \frac{\partial}{\partial p} f_i(y_1(t,p), \dots, y_n(t,p), t, p)$$

$$\Rightarrow \frac{dz_i}{dt} = \frac{\partial f_i}{\partial p} + \sum_{j=1}^{n} \frac{\partial f_i}{\partial y_j}\frac{\partial y_j}{\partial p}$$

$$\Rightarrow \boxed{\frac{d\vec{z}}{dt} = \nabla_p f + J \cdot \vec{z}.}$$

Initial Conditions:

$$z_i(0) = \lim_{\Delta p \to 0} \frac{y_i(0, p+\Delta p) - y_i(0,p)}{\Delta p}$$

1. If p is not an initial condition:
$$y_i(0, p+\Delta p) - y_i(0,p) = 0$$
for all $\Delta p \Rightarrow z_i(0) = 0$.
2. If p is an initial condition:
$$y_k(0, p+\Delta p) - y_k(0,p) = \Delta p$$
for some k. Therefore, $z_k(0) = 1$, $z_i(0) = 0$ for $i \neq k$.

Example:

1. SIR:

$$\dot{S} = -\beta S I, \quad S(0) = S_0,$$
$$\dot{I} = \beta S I - \alpha I \quad I(0) = I_0,$$
$$\dot{R} = \alpha I \quad R(0) = 0,$$

- Sensitivity of β:

$$Z_1 = \frac{\partial S}{\partial \beta}, \quad Z_2 = \frac{\partial I}{\partial \beta}, \quad Z_3 = \frac{\partial R}{\partial \beta}$$

$$\left.\begin{aligned} \dot{Z}_1 &= -SI - \beta I Z_1 - \beta S Z_2 \\ \dot{Z}_2 &= SI + \beta I Z_1 + (\beta S - \alpha) Z_2 \\ \dot{Z}_3 &= \alpha Z_2 \end{aligned}\right\} \text{S and I are computed from known solutions.}$$

$$Z_1 = Z_2 = Z_3 = 0.$$

- Sensitivity of α:

$$Z_1 = \frac{dS}{d\alpha}, \quad Z_2 = \frac{dI}{d\alpha}, \quad Z_3 = \frac{dR}{d\alpha}$$

$$\dot{Z}_1 = -\beta I Z_1 - \beta S Z_2$$
$$\dot{Z}_2 = -I + \beta I Z_1 + (\beta S - \alpha) Z_2$$
$$\dot{Z}_3 = I + \alpha Z_2$$
$$Z_1 = Z_2 = Z_3 = 0.$$

Numerical Computation for β:

$$Z_1(t+\Delta t) = Z_1(t) - \big(S(t)I(t) + \beta I(t) Z_1(t) + \beta S(t) Z_2(t)\big)\Delta t$$
$$Z_2(t+\Delta t) = Z_2(t) + \big(S(t)I(t) + \beta I(t) Z_1(t) + (\beta S(t) - \alpha) Z_2(t)\big)\Delta t$$
$$Z_3(t+\Delta t) = Z_3(t) + \alpha Z_2(t)\Delta t$$
$$Z_1(0) = Z_2(0) = Z_3(0) = 0.$$

2. SEIR:

$$\dot{S} = -\beta S I, \quad S(0) = S_0$$
$$\dot{E} = \beta S I - \eta E, \quad I(0) = I_0$$
$$\dot{I} = \eta E - \alpha I, \quad E(0) = E_0$$ ← parameter.
$$\dot{R} = \alpha I, \quad R(0) = 0$$

- Sensitivity of η:

$$z_1 = \frac{dS}{dp},\ z_2 = \frac{dE}{dp},\ z_3 = \frac{dI}{dp},\ z_4 = \frac{dR}{dp}$$

$$\Rightarrow \dot{z}_1 = -\beta I z_1 - \beta S z_3$$
$$\dot{z}_2 = -E + \beta I z_1 - \eta z_2 + \beta S z_3, \quad z_1(0) = z_2(0) = z_3(0) = z_4(0) = 0$$
$$\dot{z}_3 = E + \eta z_2 - \alpha z_3$$
$$\dot{z}_4 = \alpha z_3$$

- Sensitivity of E_0:

$$\dot{z}_1 = -\beta I z_1 - \beta S z_3$$
$$\dot{z}_2 = \beta I z_1 - \eta z_2$$
$$\dot{z}_3 = \eta z_2 - \alpha z_3$$
$$\dot{z}_4 = \alpha z_3$$
$$z_1(0) = z_3(0) = z_4(0) = 0$$
$$z_2(0) = 1.$$

Elasticity of Parameters

Sensitivity of quantity Q with respect to parameter p:

$$\Upsilon_Q^p = \frac{\partial Q}{\partial p}$$

Elasticity of quantity Q with respect to parameter p:

$$\varepsilon_Q^p = \frac{\partial Q}{\partial p}\frac{p}{Q} = \frac{\%\Delta Q}{\%\Delta p}$$

Example:

- SIR:

$$R_0 = \frac{\beta N}{\alpha}$$

$$\Rightarrow \frac{\partial R_0}{\partial \beta} = \frac{N}{\alpha}$$

$$\Rightarrow \frac{\partial R_0}{\partial \beta}\cdot\frac{\beta}{R_0} = \frac{N}{\alpha}\cdot\frac{\beta}{\beta N/\alpha} = 1$$ → 100% change in R_0 with 100% change in β.

$$\Rightarrow \frac{\partial R_0}{\partial \alpha}\cdot\frac{\alpha}{R_0} = -\frac{\beta N}{\alpha^2}\cdot\frac{\alpha}{\beta N/\alpha} = -1$$ → 100% decrease in R_0 with 100% change in α.