Lecture 12: Lyapunov Functions

Consider

$$\dot{\vec{x}} = f(x), \quad f: \mathbb{R}^n \to \mathbb{R}^n$$

with a fixed point x^*. How do we prove, if possible, all solutions converge to x^*.

Omega-limit set:

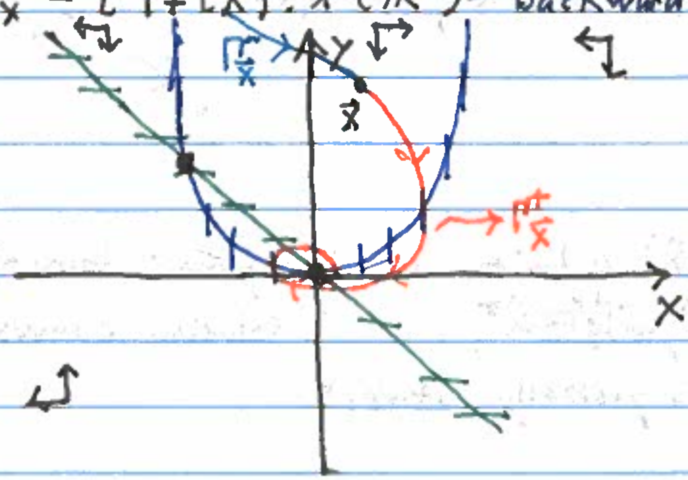
- $\varphi_t : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$

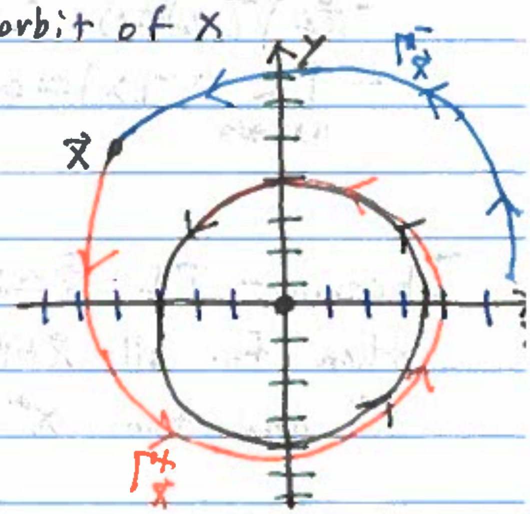
  $\varphi_t(x)$ is the value of the solution curve at time t with initial condition x.

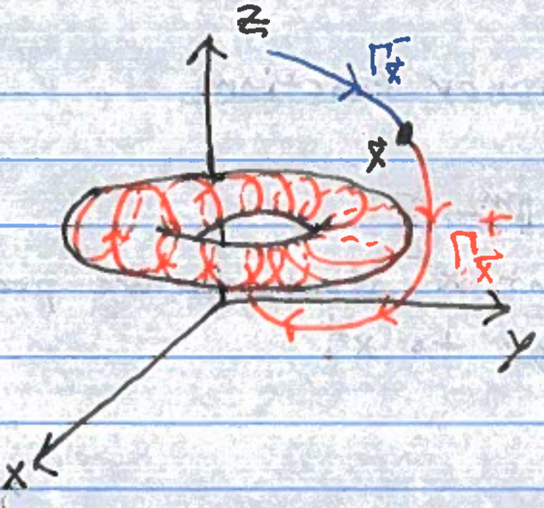
1. $\varphi_0(x) = x$
2. $\varphi_t \circ \varphi_s(x) = \varphi_t(\varphi_s(x)) = \varphi_{t+s}(x)$
3. $(\varphi_t)^{-1} = \varphi_{-t}$

$\Big\}$ group properties.

- $\Gamma_x = \{\varphi_t(x) : t \in \mathbb{R}\}$ = orbit of x
- $\Gamma_x^+ = \{\varphi_t(x) : t \in \mathbb{R}^+\}$ = forward orbit of x
- $\Gamma_x^- = \{\varphi_t(x) : t \in \mathbb{R}^-\}$ = backward orbit of x





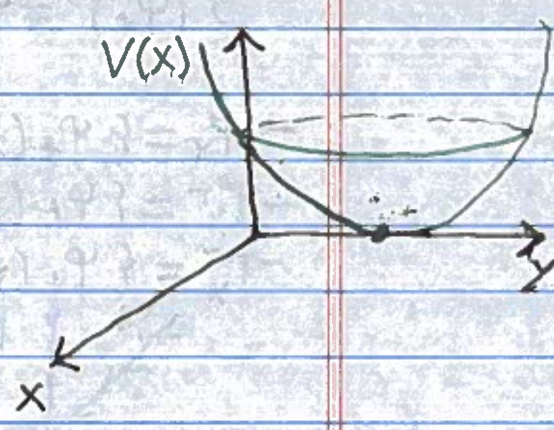


- A point $\vec{y}$ is called a <u>limit point</u> of Γ_x^+ if there exists $t_1 < t_2 < \dots < t_k < \dots$ such that $t_k \to \infty$ and $\varphi_{t_k}(x) \to y$ as $k \to \infty$.

- $\omega(\Gamma_x) =$ omega limit set $=$ the set of all limit points of Γ_x^+.

<u>Lyapunov Functions:</u>

A smooth and strong Lyapunov function is a mapping $V : \mathbb{R}^n \to \mathbb{R}$ satisfying

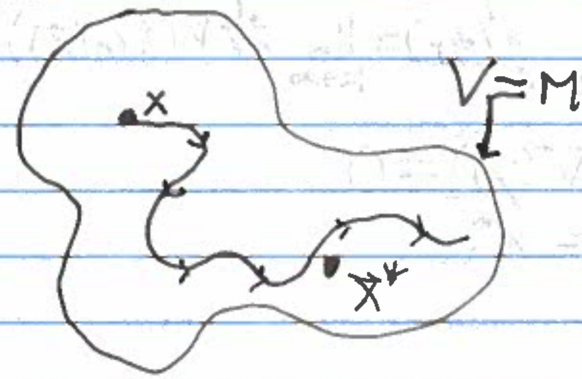


- $V \in C^\infty(\mathbb{R}^n; \mathbb{R})$, i.e. V is smooth,
- $V(x^*) = 0$,
- $V(x) > 0$ for $x \neq x^*$,
- $\frac{d}{dt} V(x(t)) < 0$ for all $x \neq x^*$, ($\dot{\vec{x}} = f(\vec{x})$)
- $\lim_{\|x\| \to \infty} V(x) = \infty$.

<u>Theorem</u> - If $\dot{\vec{x}} = f(\vec{x})$ admits a Lyapunov function then then for all $\vec{x} \in \mathbb{R}^n$, $\omega(\Gamma_{\vec{x}}^+) = \vec{x}^*$, i.e. all solution curves converge to x^* and thus x^* is globally stable.

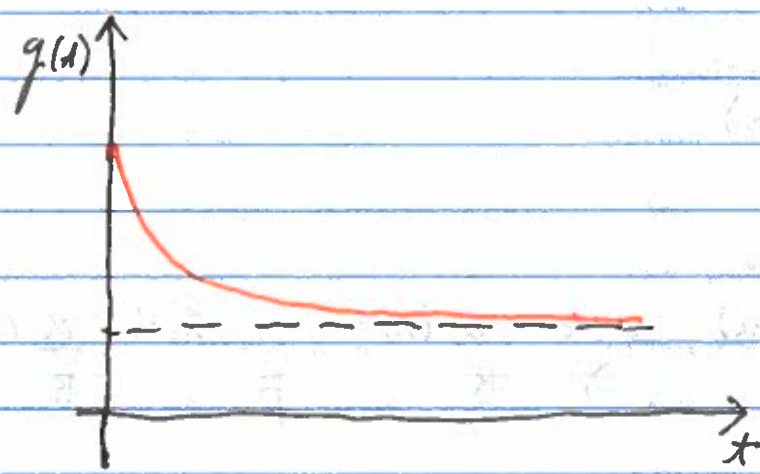
proof:

Let $\vec{x}\in\mathbb{R}^n$ satisfy $\vec{x}\neq\vec{x}^*$. Therefore, for $s>t$ it follows that $V(\varphi_s(\vec{x}))<V(\varphi_t(\vec{x}))$. Consequently, since $\lim_{\|x\|\to\infty} V(x)=\infty$ it follows that $\vec{x}(t)$ is bounded for all time.



Consequently, any limit point of Γ_x^+ must lie in $\mathbb{R}^n$, i.e. does not escape to ∞. Since $g(t)=V_t(\vec{x})$ is monotone decreasing and bounded below by 0 there exists V^* such that

$$\lim_{t\to\infty} g(t)=V^* \quad\text{and}\quad \lim_{t\to\infty} g'(t)=0.$$



However, by construction $\frac{d}{dt}V_t(x)<0$ unless $x=x^*$. Consequently, for a limit point y with a sequence of times $t_1<t_2<\dots<t_k<\dots$ satisfying $\lim_{t\to\infty}\varphi_t(x)=y$.

It follows from continuity

$$\lim_{t\to\infty} g(t) = \lim_{k\to\infty} g(t_k) = \lim_{k\to\infty} V(\varphi_{t_k}(\vec{x})) = V\left(\lim_{k\to\infty}\varphi_{t_k}(\vec{x})\right)$$

$$\Rightarrow V(y) = V(\vec{x}^*)$$

$$\lim_{t\to\infty} g'(t) = \lim_{k\to\infty} g'(t_k) = \lim_{k\to\infty} \frac{d}{dt}V(\varphi_{t_k}(\vec{x})) = \frac{d}{dt}V\left(\lim_{k\to\infty}\varphi_{t_k}(\vec{x})\right)$$

$$\Rightarrow \frac{d}{dt}(V(y)) = 0$$

Therefore, $y = x^*$.

Example:

$$\begin{aligned}
\dot{S} &= \Lambda - \beta S I - \nu S \\
\dot{E} &= \beta S I - (\nu+\gamma)E \\
\dot{I} &= \gamma E - (\nu+\alpha) I \\
\dot{R} &= \alpha I - \nu R
\end{aligned}
\quad , \quad R_0 = \frac{\Lambda \beta \gamma}{(\nu+\gamma)(\nu+\alpha)\nu}$$

Disease free equilibrium:

$$\vec{x}_1^* = (\Lambda/\nu, 0, 0, 0)$$

Endemic equilibrium:

$$\vec{x}_2^* = \left(\frac{(\nu+\gamma)(\nu+\alpha)}{\beta\gamma}, \frac{\nu+\alpha}{\gamma}\frac{\nu}{\beta}(R_0-1), \frac{\nu}{\beta}(R_0-1), \frac{\alpha}{\beta}(R_0-1)\right)$$

Consider

$$V = K\left(S - \frac{\Lambda}{\nu} - \frac{\Lambda}{\nu}\ln\left(\frac{\nu S}{\Lambda}\right)\right) + \frac{1}{\nu+\gamma}E + \frac{1}{\gamma}I$$

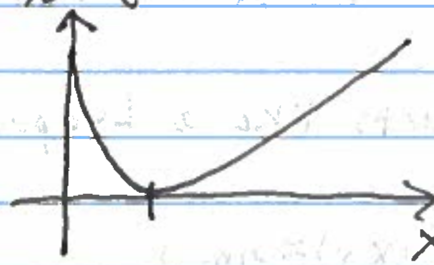
$- V(\Delta/\nu, 0, 0, 0) = 0$

$- S - \Delta/\nu - \Delta/\nu \ln\left(\frac{\nu S}{\Delta}\right) = \frac{\Delta}{\nu}\left(\frac{\nu}{\Delta} S - 1 - \ln\left(\frac{\nu S}{\Delta}\right)\right)$

Let $g(x) = x - 1 - \ln(x)$.

$g'(x) = 1 - 1/x = 0$

$\Rightarrow x = 1$



Therefore, $V > 0$ except at the endemic equilibrium.

$$\frac{dV}{dt} = \nabla V \cdot \dot{\vec{x}} = \nabla V \cdot \vec{f}$$

$$\nabla V = \begin{bmatrix} K(1 - \Delta/\nu \cdot \frac{1}{S}) \\ 1/(\nu+\gamma) \\ 1/\gamma \end{bmatrix}$$

$$\Rightarrow \nabla V \cdot \vec{f} = K(1 - \Delta/\nu \cdot 1/S)(\Delta - \beta S I - \nu S) + (\beta S I - (\nu+\gamma)E)\, 1/(\nu+\gamma) + (\gamma E - (\nu+\alpha) I)\, 1/\gamma$$

$$= 2K\Delta - K\beta S I - K\nu S - K\Delta^2/\nu S + \frac{K\Delta\beta I}{\nu} + \beta S I/(\nu+\gamma) - (\nu+\alpha)/\gamma\, I$$

Pick $K = 1/(\nu+\gamma)$ we have:

$$\nabla V \cdot \vec{f} = -K\Delta\left(\frac{\Delta}{\nu S} + \frac{\nu S}{\Delta} - 2\right) + \frac{\Delta\beta}{\nu(\nu+\gamma)} I - \frac{(\nu+\alpha)}{\gamma} I$$

$$= -K\Delta\left(\frac{\Delta}{\nu S} + \frac{\nu S}{\Delta} - 2\right) + \frac{\nu+\alpha}{\gamma}\left(R_0 - 1\right) I$$

Need to prove this is negative if $R_0 < 1$. Now

$$x + \frac{1}{x} - 2 = \frac{x^2 - 2x + 1}{x} = \frac{(x-1)^2}{x} > 0.$$

Therefore, if $R_0 < 1$ it follows that $\frac{dV}{dt} < 0 \Rightarrow$ Lyapunov function. ■

Example:

$\dot{x} = -\nabla V$

$\frac{dV}{dt} = -\nabla V \cdot \dot{x} = -\|\nabla V\|^2 \le 0$

V acts like a Lyapunov function.

$\dot{x} = f(x,y) = \sin(y)$

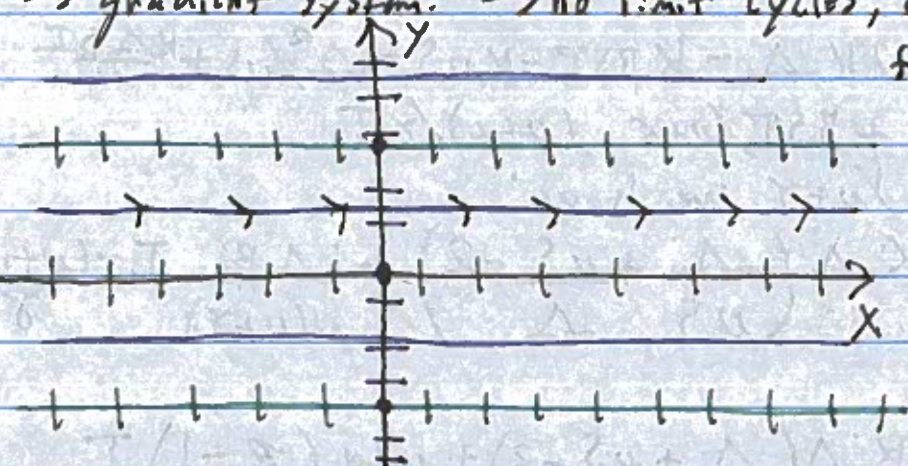
$\dot{y} = g(x,y) = x\cos(y)$

If $\dot{x} = f(x,y) = \frac{\partial V}{\partial x}$, $\dot{y} = g(x,y) = \frac{-\partial V}{\partial y}$

$\Rightarrow \frac{\partial f}{\partial y} = \frac{\partial f}{\partial x}$

$\frac{\partial f}{\partial y} = \cos(y)$, $\frac{\partial f}{\partial x} = \cos(y)$, $V = x\sin(y)$.

→ gradient system. ⇒ no limit cycles, all trajectories go to fixed points.



Example:

$$\ddot{\vec{x}} = -\nabla V(\vec{x})$$

$$\Rightarrow \dot{\vec{x}} \cdot \ddot{\vec{x}} = -\dot{\vec{x}}\, \nabla V(\vec{x})$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|\dot{\vec{x}}\|^2 = -\frac{d}{dt}V(\vec{x})$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}\|\dot{\vec{x}}\|^2 + V(\vec{x})\right) = 0$$

$\Rightarrow \frac{1}{2}\|\dot{x}\|^2 + V(x)$ is conserved

$$\dot{\vec{x}} = \vec{v}$$

$$\dot{\vec{v}} = -\nabla V(x)$$

$\Rightarrow \frac{1}{2}\|v\|^2 + V(x)$ is conserved.

Example:

$$\ddot{\vec{x}} + \dot{\vec{x}}^3 = -\nabla V(x)$$

$$\Rightarrow \dot{\vec{x}}\, \ddot{\vec{x}} + \dot{\vec{x}}^4 = -\dot{x}\, \nabla V(x).$$