Lecture 2: SIS Model

$$S \underset{\alpha}{\overset{\beta I}{\rightleftarrows}} I$$

Simple Model (Analytic Approach)

$$\dot{S} = -\beta I S + \alpha I$$
$$\dot{I} = \beta I S - \alpha I$$

$\Rightarrow \dot{N} = \dot{S} + \dot{I} = 0$

$\Rightarrow N = S + I = S_0 + I_0$ is constant in time.

Therefore,

$$\dot{I} = \beta I (N - I) - \alpha I$$

Let

$$x = I/N, \quad \tau = \alpha t$$

$$\Rightarrow \dot{I} = \frac{dI}{dt} = N\frac{dx}{dt} = N\frac{dx}{d\tau}\frac{d\tau}{dt} = N\alpha\frac{dx}{d\tau}$$

$$\Rightarrow N\alpha\frac{dx}{d\tau} = \beta N^2 x(1-x) - \alpha N x$$

$$\Rightarrow \frac{dx}{d\tau} = R_0 x(1-x) - x$$

$$= x((R_0 - 1) - x)$$

$$\Rightarrow \int_{x_0}^{x} \frac{1}{x(R_0-1-x)}\,dx = \int_0^t dt$$

$$\Rightarrow \frac{1}{R_0-1}\int_{x_0}^{x}\left(\frac{1}{x} + \frac{1}{R_0-1-x}\right)dx = t$$

$$\Rightarrow \frac{1}{R_0-1} \ln\left(\frac{x}{x_0}\right) - \ln\left(\frac{R_0-1-x}{R_0-1-x_0}\right) = t$$

$$\Rightarrow \ln\left(\frac{x}{x_0}\,\frac{R_0-1-x_0}{R_0-1-x}\right) = (R_0-1)t$$

$$\Rightarrow \frac{x}{x_0}\,\frac{R_0-1-x_0}{R_0-1-x} = \exp((R_0-1)t)$$

$$\Rightarrow x\,(R_0-1-x_0) = x_0\,(R_0-1-x)\exp((R_0-1)t)$$

$$\Rightarrow x\,(R_0-1-x_0+x_0\exp((R_0-1)t)) = x_0(R_0-1)\exp((R_0-1)t).$$

$$\Rightarrow \boxed{x(t) = \frac{x_0(R_0-1)\exp((R_0-1)t)}{R_0-1-x_0(1-\exp((R_0-1)t))}}$$

If $R_0 > 1$ it follows that

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{x_0(R_0-1)^2\exp((R_0-1)t)}{x_0(R_0-1)\exp((R_0-1)t)} = R_0-1.$$

In this case we say the disease becomes <u>endemic</u>.

If $R_0 < 1$ it follows that

$$\lim_{t\to\infty} x(t) = \frac{0}{R_0-1} = 0.$$

In this case the treatment rate is high enough that the disease is eradicated.

Simple Model (Graphical Approach)

$$\frac{dx}{d\tau} = R_0 x(1-x-1/R_0)$$

$$= R_0 x\left(\frac{R_0-1}{R_0} - x\right)$$

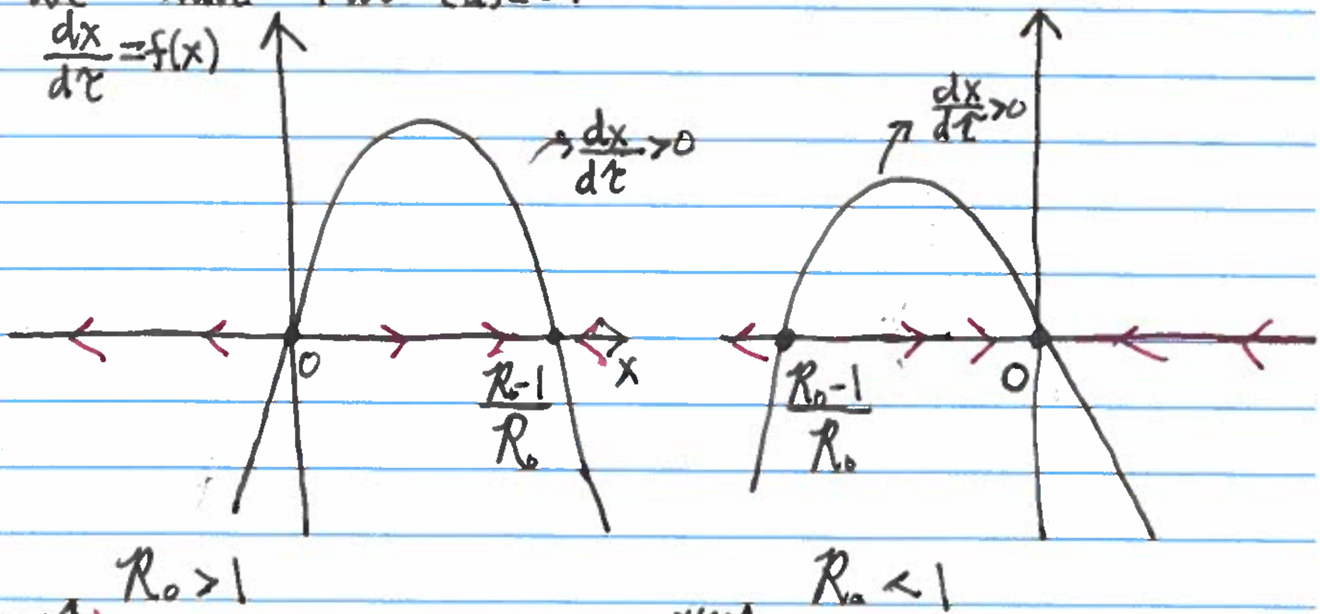
$$x_1^* = 0$$

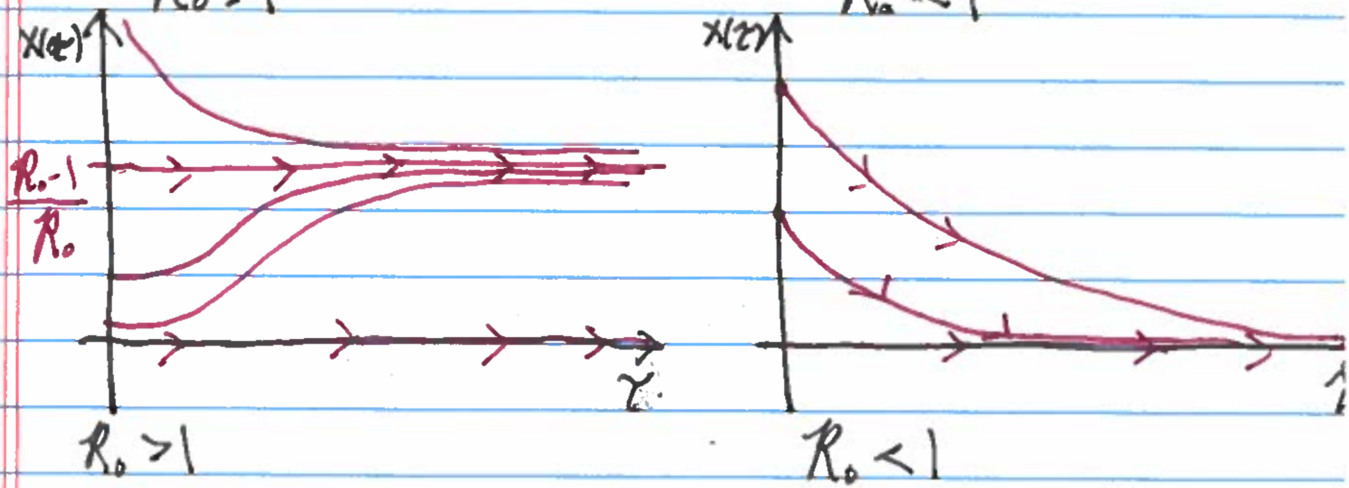
$$x_2^* = \frac{R_0-1}{R_0}$$

Letting

$$f(x) = R_0 x\left(\frac{R_0 - 1}{R_0} - x\right),$$

we have two cases:





Inflection points occur when $\frac{d^2x}{d\tau^2} = 0$. Calculating it follows that

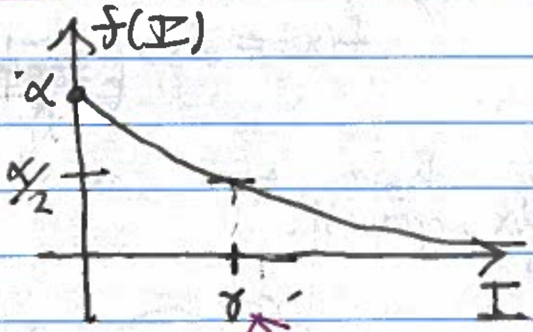
$$\frac{d^2x}{d\tau^2} = \frac{d}{d\tau}\frac{dx}{d\tau} = \frac{d}{d\tau} f(x) = f'(x)\frac{dx}{d\tau} = f'(x)f(x).$$

Inflection points occur when $f'(x) = 0$.

$$\Rightarrow x = \frac{R_0 - 1}{2R_0}.$$

SIS Model with Saturating Treatment

$$\dot{S} = -\beta I S + f(I) I$$
$$\dot{I} = \beta I S - f(I) I$$



Capacity when treatment rate is half effective.

Simplest form:
$$f(I) = \frac{\alpha}{1 + I/\gamma}$$

$$\Rightarrow \dot{S} = -\beta I S + \frac{\alpha I}{1 + I/\gamma}$$
$$\dot{I} = \beta I S - \frac{\alpha I}{1 + I/\gamma}$$

$$\Rightarrow \dot{I} = I\left(\beta(N - I) - \frac{\alpha I}{1 + I/\gamma}\right)$$

Let $x = I/N$, $\tau = \alpha t$.
$$\Rightarrow N\alpha \frac{dx}{d\tau} = N x\left(\beta N(1-x) - \frac{\alpha x}{1 + Nx/\gamma}\right)$$
$$\Rightarrow \boxed{\frac{dx}{d\tau} = x R_0 (1-x) - \frac{x}{1+sx} = f(x).}$$

Fixed points:
$$0 = x\left(R_0(1-x) - \frac{1}{1+sx}\right)$$
$$\Rightarrow 0 = x\left(\frac{R_0(1-x)(1+sx) - 1}{1+sx}\right)$$

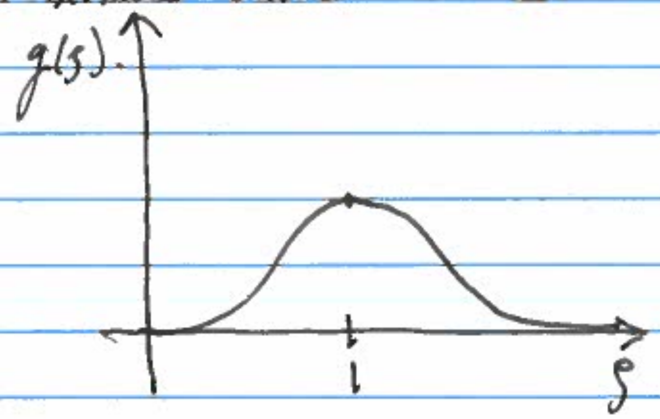
$$\Rightarrow R_0(1-\varsigma)^2 > 4\varsigma(1-R_0)$$
$$\Rightarrow R_0[(1-\varsigma)^2+4\varsigma] > 4\varsigma$$
$$\Rightarrow R_0[(1+\varsigma)^2] > 4\varsigma$$
$$\Rightarrow R_0 > \frac{4\varsigma}{(1+\varsigma)^2}$$

Additional fixed points exist if $R_0 > 4\varsigma/(1+\varsigma)^2$. Moreover,
$$\lim_{x\to\infty} f(x) = -\infty.$$

If we define
$$g(\varsigma) = \frac{4\varsigma}{(1+\varsigma)^2}$$

it follows that
$$g'(\varsigma) = \frac{4(1+\varsigma)^2 - 4\varsigma\cdot 2(1+\varsigma)}{(1+\varsigma)^2}$$
$$\Rightarrow g'(\varsigma) = \frac{4(\varsigma^2-1)}{(1+\varsigma)^2}$$

Therefore, g has a critical point at $\varsigma=1$ and thus a maximum value of 1:



$$\Rightarrow x=0, \quad R_0(1-x)(1+\rho x)-1=0$$

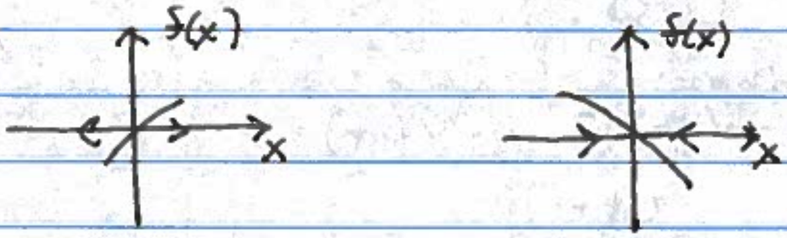
$$\Rightarrow x=0, \quad (x-1)(\rho x+1)+1/R_0=0$$

$$\Rightarrow x=0, \quad \rho x^2+(1-\rho)x+1/R_0-1=0$$

$$\Rightarrow x=0, \quad \rho x^2+(1-\rho)x+\frac{1-R_0}{R_0}=0.$$

$$\Rightarrow x=0, \quad \boxed{x^*_{\pm}=\frac{\rho-1\pm\sqrt{(1-\rho)^2-4\rho(1-R_0)/R_0}}{2\rho}.} \ *$$

The $x=0$ case always exists. Lets analyze its stability. We just need to check the slope at $x=0$.



unstable stable

$$f'(x)=R_0(1-2x)-\frac{[(1+\rho x)-\rho x]}{(1+\rho x)^2}$$

$$\Rightarrow f'(0)=R_0-1$$

Therefore,

a) If $R_0>1 \Rightarrow 0$ is unstable

b.) If $R_0<1 \Rightarrow 0$ is stable.

Now, we check the existence of other fixed points

$$\Rightarrow (1-\rho)^2-\frac{4\rho(1-R_0)}{R_0}>0$$

$$\Rightarrow (1-\rho)^2>\frac{4\rho(1-R_0)}{R_0}$$

Therefore, we have cases to consider.

Case 1:

If $R_0 < \frac{4\varsigma}{(1+\varsigma)^2} \Rightarrow x_1^*$ is the only fixed point and is stable. (Disease eliminated)

Case 2:

If $\frac{4\varsigma}{1+\varsigma^2} < R_0 < 1$, $\varsigma < 1 \Rightarrow x_1^*$ is stable, x_-^* is negative, x_+^* is negative

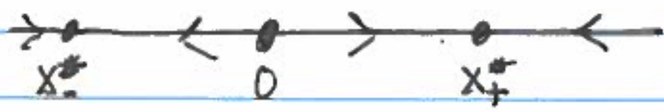
Case 3:

If $\frac{4\varsigma}{1+\varsigma^2} < R_0 < 1$, $\varsigma > 1 \Rightarrow x_1^*$ is stable, x_-^*, x_+^* are both positive.

(This is the dangerous regime)

Case 4:

$R_0 > 1 \Rightarrow x_1^*$ is unstable, x_-^* is negative, x_+^* is positive.



We can put this together into a phase diagram.