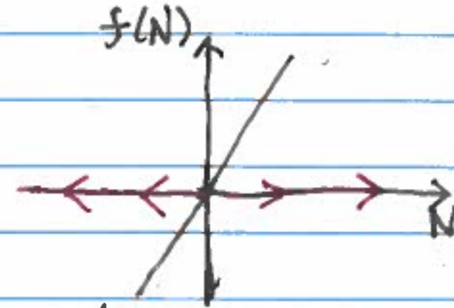
Lecture 3: SIR with Demography.

Population Growth.

Model 1

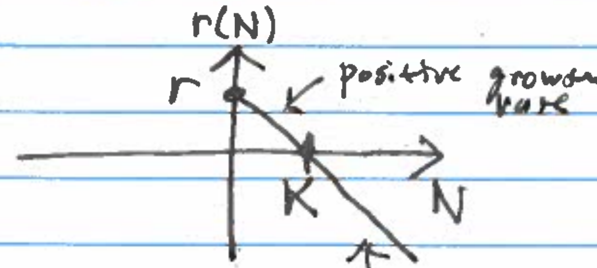


$$\dot{N} = bN - \mu N = rN = f(N)$$

birth rate, death rate, growth rate.

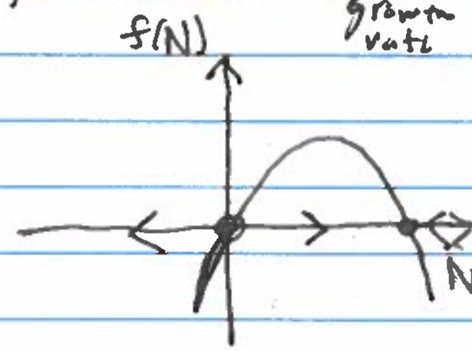
$N^* = 0$ is unstable, $x(t) = x_0 e^{-rt}$.

Model 2:



$$\dot{N} = r(N)N$$

growth rate function of population size.



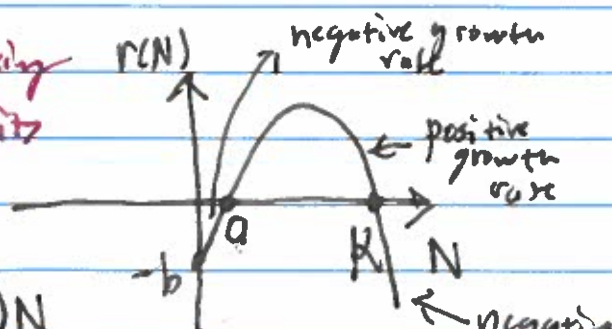
$$\Rightarrow \dot{N} = r(1 - N/K)N$$

$N^* = 0$ (unstable)

$N^* = K$ (stable)

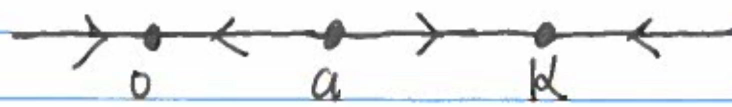
↳ carrying capacity

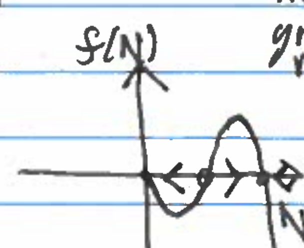
Model 3:



$$\dot{N} = r(N)N$$

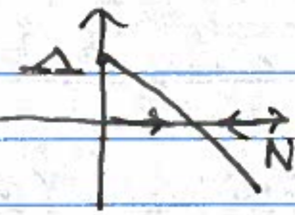
$$\Rightarrow \dot{N} = -b(1 - N/a)(1 - N/K)N$$





Model 4':

$$\dot{N} = \Lambda - \nu N = f(N)$$



↑constant birth rate

$\Rightarrow N = \nu/\Lambda$ is a stable fixed point.

(Commonly used in epidemiology for its simplicity)

SIR with Demography

$$\dot{S} = \Lambda - \beta I S - \nu S$$
$$\dot{I} = \beta I S - \alpha I - \nu I$$
$$\dot{R} = \alpha I - \nu R$$

$$N = I + S + R$$
$$\dot{N} = \dot{I} + \dot{S} + \dot{R}$$
$$= \Lambda - \nu(S + I + R)$$
$$= \Lambda - \nu N.$$

(Non constant population!)

Since R is decoupled we have that

$$R = N - I - S$$
$$= \frac{\Lambda}{\nu} + Ce^{-\nu t} - I - S$$

$$\Rightarrow \dot{S} = \Lambda - \beta I S - \nu S$$
$$\dot{I} = \beta I S - \alpha I - \nu I.$$

$[\Lambda] = $ pop/time

$[\beta] = 1/$pop$\cdot$time.

$[\nu] = 1/$time

$[\alpha] = 1/$time.

$\tau = \alpha t$

$x = \dfrac{\nu}{\Lambda} S$ → 1/equilibrium population

$y = \dfrac{\nu}{\Lambda} I$

$$\alpha \frac{\Lambda}{\nu} \frac{dx}{d\tau} = \Lambda - \beta \frac{\Lambda^2}{\nu^2} xy - \nu \frac{\Lambda}{\nu} x$$

$$\alpha \frac{\Lambda}{\nu} \frac{dI}{d\tau} = \beta \frac{\Lambda^2}{\nu^2} xy - \alpha \frac{\Lambda}{\nu} y - \nu \frac{\Lambda}{\nu} y$$

$$\Rightarrow \frac{dx}{d\tau} = \frac{\nu}{\alpha} - \beta \frac{\Lambda}{\nu\alpha} xy - \frac{\nu}{\alpha} x$$

$$\frac{dy}{d\tau} = \beta \frac{\Lambda}{\nu\alpha} xy - y - \frac{\nu}{\alpha} y$$

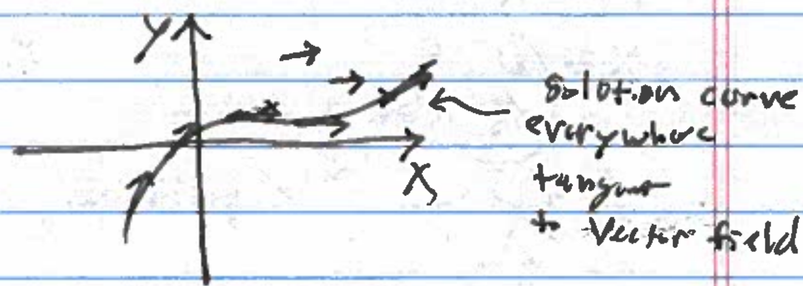
Letting $A = \beta \Lambda / \nu\alpha$, $\xi = \nu/\alpha$ we obtain:

$$\boxed{\begin{aligned} \frac{dx}{d\tau} &= \xi(1-x) - Axy \\ \frac{dy}{d\tau} &= y(Ax - (1+\xi)) \end{aligned}}$$

Analysis of System

$\dot{\vec{x}} = F(\vec{x})$

tangent vector to curve — vector field



$$\frac{dx}{d\tau} = \xi(1-x) - Axy$$

$$\frac{dy}{d\tau} = y(Ax-1-\xi)$$

Fixed Points:

$$\xi(1-x) - Axy = 0$$

$$y(Ax-1-\xi) = 0$$

Fixed Point #1.

$y_1^* = 0, \; x_1^* = 1.$ (Disease Free State)

Fixed Point #2:

$$x_1^* = \frac{1+\xi}{A}, \quad \xi\left(1 - \frac{(1+\xi)}{A}\right) - A\frac{(1+\xi)}{A}y = 0$$

$$\Rightarrow x_2^* = \frac{1+\xi}{A}, \quad y_2^* = \frac{\xi}{1+\xi}\left(1 - \frac{(1+\xi)}{A}\right)$$ (Endemic Equilibrium).

Nullclines:

N1:

$$\frac{dx}{d\tau} = 0 \Rightarrow y = \frac{s(1-x)}{Ax}$$

N2:

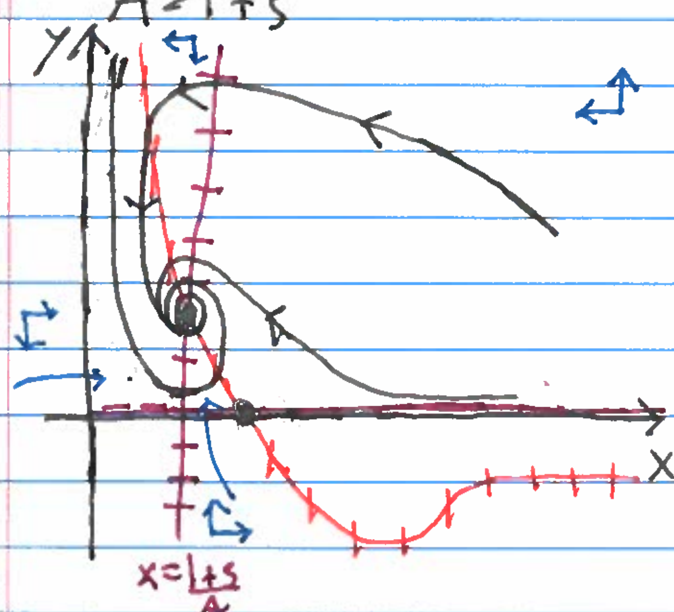
$$\frac{dy}{d\tau} = 0 \Rightarrow y = 0$$

N3:

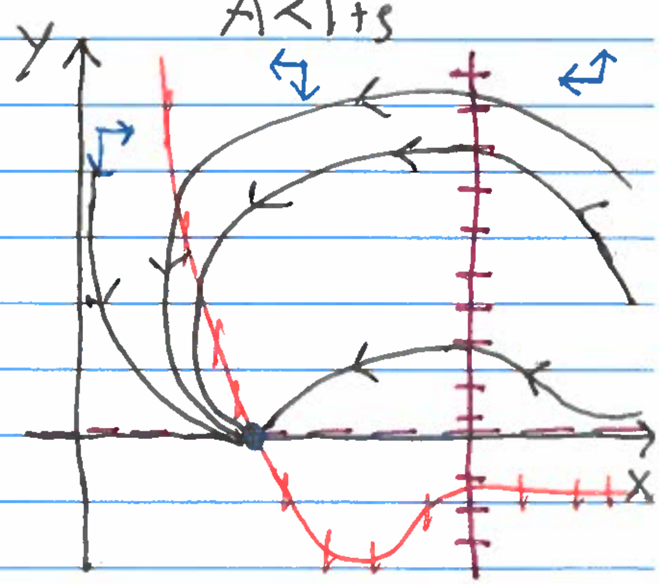
$$\frac{dy}{d\tau} = 0 \Rightarrow x = \frac{1+s}{A}$$

Case 1:

$A > 1+s$



$A < 1+s$



Endemic State is stable!

Disease free State is stable!

$$\Rightarrow R_0 = \frac{A}{1+s} = \frac{\beta\Delta/\nu\alpha}{1+\nu/\alpha} = \frac{\beta\Delta}{\nu(\alpha+\nu)}.$$