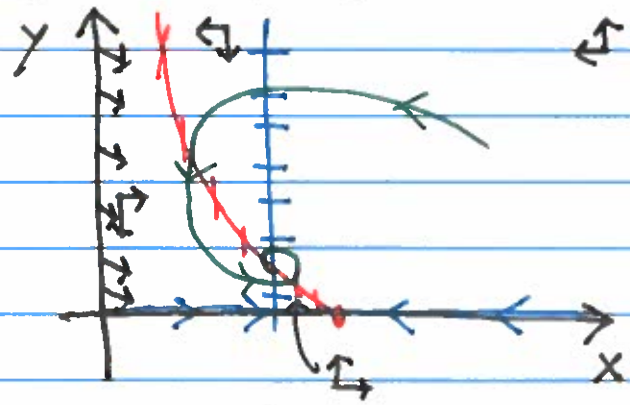
Lecture 4: Stability Analysis

$$\frac{dx}{d\tau} = \varsigma(1-x) - Axy$$

$$\frac{dy}{d\tau} = y(Ax - (1+\varsigma))$$

Case $A > 1+\varsigma$:



1. If $y = 0$:

$$\Rightarrow \frac{dy}{d\tau} = 0 \quad \text{and} \quad \frac{dx}{d\tau} = \varsigma(1-x)$$

$\downarrow$

Invariant Submanifold

Dynamics stays on $y=0$

$\searrow u = x - 1$ ← Change coordinate

$$\frac{du}{d\tau} = \frac{dx}{d\tau} = -\varsigma u$$

$$\Rightarrow u(\tau) = u_0 \exp(-\varsigma t)$$

$$\Rightarrow x(\tau) - 1 = (x_0 - 1)\exp(-\varsigma t)$$

$$\Rightarrow x(\tau) = 1 + (x_0 - 1)\exp(-\varsigma t)$$

$$\Rightarrow \lim_{t\to\infty} x(\tau) = 1$$

$$\Rightarrow \lim_{t\to\infty} S(t) = \frac{\Lambda}{\mathcal{N}}$$ ← carrying capacity

2. If $x=0$:

$$\frac{dx}{d\tau}=\rho\,,\quad \frac{dy}{d\tau}=-y(1+\rho)$$

$\Rightarrow$ Everything on the line $x=0$ moves to the right

$\Rightarrow$ 1st quadrant is an invariant manifold.

3. Near $(1,0)$:

$$\frac{dx}{d\tau}=f(x,y)=\rho(1-x)-Axy$$

$$\frac{dy}{d\tau}=g(x,y)=y(Ax-(1+\rho))$$

Taylor Expand:

$$f(x,y)=f(1,0)+\frac{\partial f}{\partial x}\Big|_{(1,0)}(x-1)+\frac{\partial f}{\partial y}\Big|_{(1,0)}y+\text{H.O.T.s.}$$

$$=0+(-\rho-Ay)\Big|_{(1,0)}(x-1)+(-Ax)\Big|_{(1,0)}y+\text{HOTS.}$$

$$=-\rho(x-1)-Ay+\text{HOTS}$$

$$g(x,y)=g(1,0)+\frac{\partial g}{\partial x}\Big|_{(1,0)}(x-1)+\frac{\partial g}{\partial y}\Big|_{(1,0)}y+\text{HOTS.}$$

$$=(Ay)\Big|_{(1,0)}(x-1)+(Ax-(1+\rho))\Big|_{(1,0)}y+\text{HOTS}$$

$$=(A-(1+\rho))y$$

$$\Rightarrow \frac{d\vec{x}}{d\tau}\approx\begin{bmatrix}\frac{dx}{d\tau}\\ \frac{dy}{d\tau}\end{bmatrix}=\begin{bmatrix}-\rho(x-1) & -Ay\\ 0 & +(A-(1+\rho))y\end{bmatrix}=\begin{bmatrix}-\rho & -A\\ 0 & A-(1+\rho)\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

$$\Rightarrow \frac{d\vec{x}}{d\tau} \approx J(1,0)\,\vec{x},$$

where the Jacobian matrix $J(x,y)$ is given by:

$$J(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}.$$

Big picture:

Near a fixed point $\vec{x}^*$:

$$\frac{d\vec{x}}{d\tau} \approx J(\vec{x}^*)(\vec{x} - \vec{x}^*)$$

Let $\vec{v} = \vec{x} - \vec{x}^*$ $\longrightarrow$ measures separation from fixed point.

$$\Rightarrow \boxed{\frac{d\vec{v}}{d\tau} \approx J(\vec{x}^*)\,\vec{v}} \quad (*)$$

Linear ODE

Guess solution:

$$\vec{v} = e^{\lambda_1 t}\vec{v}_1,$$

where λ_1 is an eigenvalue of $J(x^*)$ with corresponding eigenvector $\vec{v}_1$

$$\frac{d\vec{v}}{dt} = \lambda_1 e^{\lambda_1 t}\vec{v}_1$$

$$= e^{\lambda_1 t}\lambda\vec{v}_1$$

$$= e^{\lambda_1 t} J(\vec{x}^*)\vec{v}_1$$

$$= J(\vec{x}^*)(e^{\lambda_1 t}\vec{v}_1)$$

$$= J(\vec{x}^*)\,\vec{v}.$$

This works!

By linearity it follows that generic solution to $(*)$ is of the form

$$\boxed{\vec{v}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2}$$

Therefore,

$$\lim_{t\to\infty} u(t) = 0 \text{ if and only if } Re(\lambda_1, \lambda_2) < 0!$$

Return to epidemiology:

$$J(1,0) = \begin{bmatrix} -\rho & -A \\ 0 & A-(1+\rho) \end{bmatrix}$$

$$\Rightarrow \lambda_1 = -\rho$$
$$\lambda_2 = A-(1+\rho)$$

$\Rightarrow (1,0)$ is stable if $A-(1+\rho) < 0$

4. Near $\left(\frac{1+\rho}{A}, \frac{\rho}{(1+\rho)A}(A-(1+\rho))\right)$:

$$J(x,y) = \begin{bmatrix} -\rho - Ay & -Ax \\ Ay & Ax-(1+\rho) \end{bmatrix}$$

$$J\left(\frac{1+\rho}{A}, \frac{\rho}{(1+\rho)A}(A-(1+\rho))\right) = \begin{bmatrix} -\rho - \frac{\rho}{1+\rho}(A-(1+\rho)) & -(1+\rho) \\ \frac{\rho}{1+\rho}(A-(1+\rho)) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{\rho A}{1+\rho} & -(1+\rho) \\ \frac{\rho}{1+\rho}(A-(1+\rho)) & 0 \end{bmatrix}$$

Therefore,

$$Tr(J(x_2^*)) = \frac{-sA}{1+s} = \lambda_1 + \lambda_2$$

$$Det(J(x_2^*)) = s(A-(1+s)) = \lambda_1 \lambda_2$$

$$\Rightarrow \lambda_2 = Tr(J(x_2^*)) - \lambda_1$$

$$\lambda_1(Tr(J(x_2^*)) - \lambda_1) = Det(J(x_2^*))$$

$$\Rightarrow \lambda_1^2 - Tr(J(x_2^*))\lambda_1 + Det(J(x_2^*)) = 0$$

$$\Rightarrow \lambda_1 = \frac{Tr \pm \sqrt{Tr^2 - 4Det}}{2}$$

If fixed point is stable need $Tr < 0$ and $Det > 0$.

proof:

1. If $Tr > 0$ then $Re(Tr + \sqrt{Tr^2 - 4Det}) > 0$.
2. If $Tr < 0$ and $Det < 0$ then

$$\sqrt{Tr^2 - 4Det} > Tr$$

$$\rightarrow Tr + \sqrt{Tr^2 - 4Det} > 0.$$

In our problem

$$\boxed{\begin{array}{l} Tr(J(x_2^*)) = \dfrac{-sA}{1+s} < 0 \\ Det(A-(1+s)) > 0 \Rightarrow A > 1+s. \end{array}}$$