MTH 317/617
Fall 2022
Exam 1
09/23/22

Name (Print): Key

This exam contains 11 pages (including this cover page) and 12 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You are required to show your work on each problem on this exam. The following rules apply:

- **Resources:** You are allowed to use your text, notes, and a calculator. You are not allowed to use the internet, software or any other external resource.
- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Undergraduate Problems:** Questions labeled as **"Undergraduate problems"** will only count for undergraduate students. Graduate students do not have to complete these problems.
- **Graduate Problems:** Questions labeled as **"Graduate problems"** must be completed by the graduate students to receive credit. Undergraduate students can complete these problems for extra credit.
- **Short answer questions:** Questions labeled as **"Short Answer"** can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit.** An incorrect answer supported by substantially correct calculations and explanations might receive partial credit.

Problem	Points	Score
1	10	
2	10	
3	10	
4	15	
5	15	
6	10	
7	5	
8	10	
9	5	
10	0	
11	10	
12	0	
Total:	100	

Do not write in the table to the right.

1. (10 points) **(Short Answer)** Determine if the following statement is correct (**C**) or incorrect (**I**). Just circle **C** or **I**. No need to show any work. In order for a statement to be correct it must be true in all cases.

   C **I** If $z_1, z_2 \in \mathbb{C}$ then $\text{Re}(z_1 z_2) = \text{Re}(z_1)\text{Re}(z_2)$.

   C **I** $z \in \mathbb{C}$ then z^2 is a real number.

   C **I** If $z \in \mathbb{C}$ then $|e^{iz}| = 1$.

   C **I** If $z \in \mathbb{C}$ satisfies $e^z = 1$ then $z = 0$.

   C **I** If $f : \mathbb{C} \mapsto \mathbb{C}$ is a polynomial and $f(z) = i$ then $f(\bar{z}) = -i$.

2. (10 points) Convert the following complex numbers $z \in \mathbb{C}$ into exponential form, i.e. $z = re^{i\theta}$.

   (a) (5 points) $z = -i$

   $z = e^{-i\pi/2}$

   (b) (5 points) $z = -1 + i$

   $z = \sqrt{2}\, e^{3\pi i/4}$

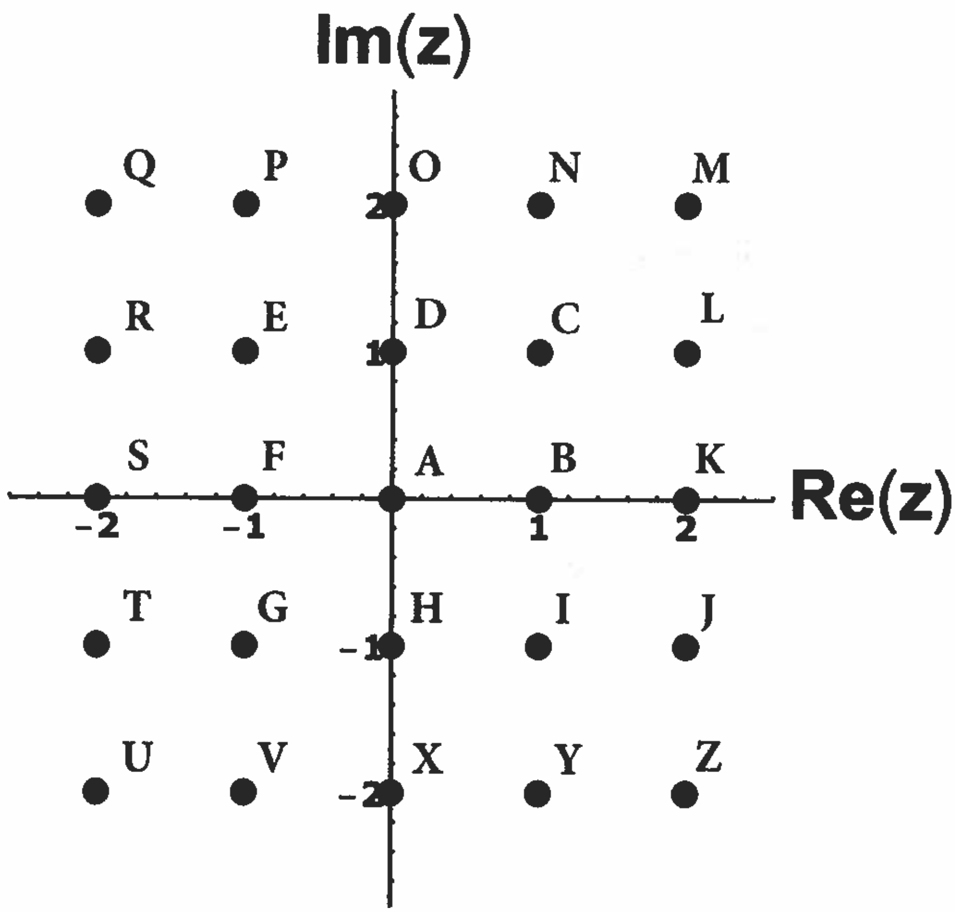
3. (10 points) (**Short Answer**) Match the following complex numbers with the points drawn below on the complex plane $\mathbb{C}$. If no points drawn correspond to the value of z, simply write **no solution**.

   (a) (2.5 points) $z = 2$ – K

   (b) (2.5 points) $z = 1 - i$ – I

   (c) (2.5 points) $z = 2\sqrt{2}e^{\frac{3\pi i}{4}}$ – Q

   (d) (2.5 points) $z = e^{0}$ – B



4. (15 points) Find all solutions to the following equations for $z \in \mathbb{C}$. You can leave your solutions in Cartesian or exponential form.

(a) (5 points) $\dfrac{1-z}{1+z} = 2i$

$1 - z = 2i(1+z)$

$1 - 2i = z(1+2i)$

$\Rightarrow z = \dfrac{1-2i}{1+2i}$

(b) (10 points) $z^3 = -i$.

$z = \left(e^{-i\pi/2 + 2n\pi i}\right)^{1/3}$,

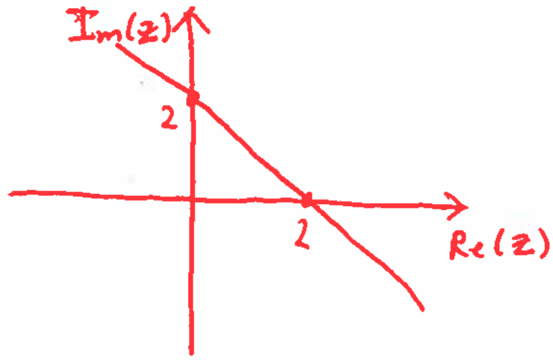
$= e^{-i\pi/6 + 2n\pi i/3}$,

where $n \in \mathbb{Z}$.

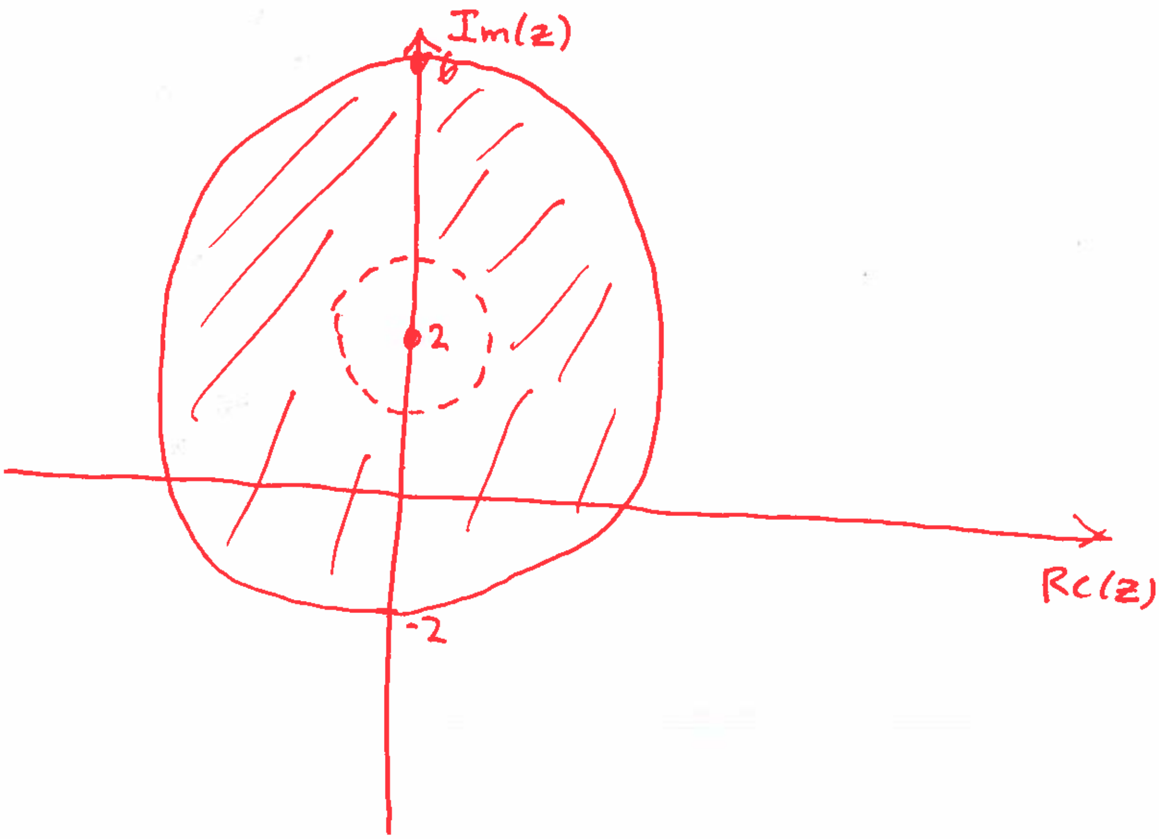
5. (15 points) (**Short Answer**) Plot the set of points $z \in \mathbb{C}$ satisfying the equations and inequalities below. For each problem, make sure to sketch the real and imaginary axis and carefully label your graph. Also, use the standard convention of dashed and solid curves to indicate if the boundary of the set is open or closed.

(a) (5 points) $\mathrm{Re}(z) + \mathrm{Im}(z) = 2$

$\mathrm{Im}(z) = 2 - \mathrm{Re}(z)$

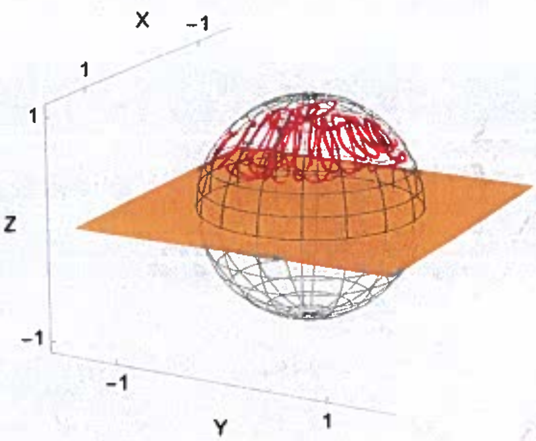


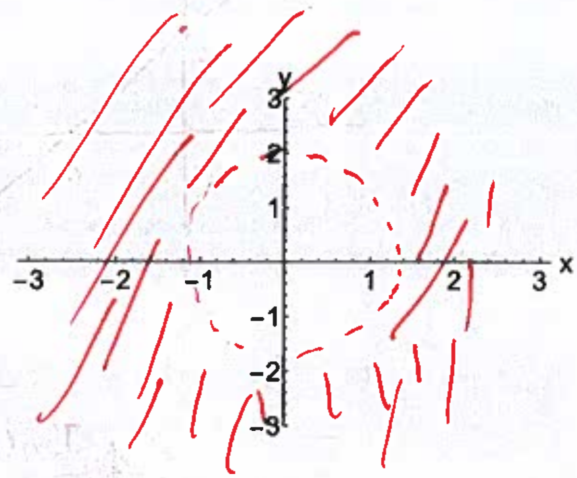
(b) (10 points) $1 < |z - 2i| \leq 4$



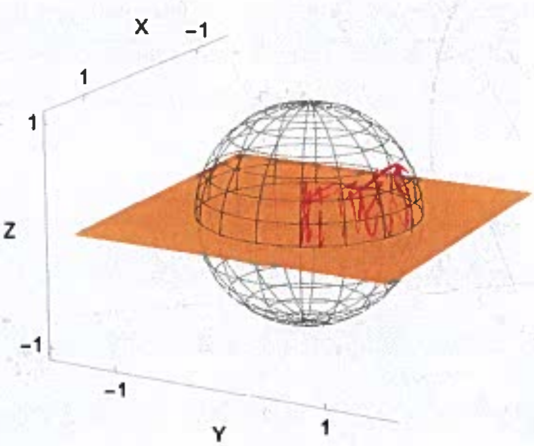
6. (10 points) (**Short Answer**) In the following figures, the Riemann sphere is plotted as a wire mesh with the complex plane embedded through its equator. The coordinates X, Y, Z denote coordinates in three dimensional space. In each problem, shade in the subset S on the Riemann sphere and then plot the stereographic projection of S onto $\mathbb{C}$ on the x-y axes provided. **Do not fuss over getting the perfect plots. You just need to demonstrate understanding of the problem.** Again, use the standard convention of dashed and solid curves to indicate if the boundary of the set is open or closed.

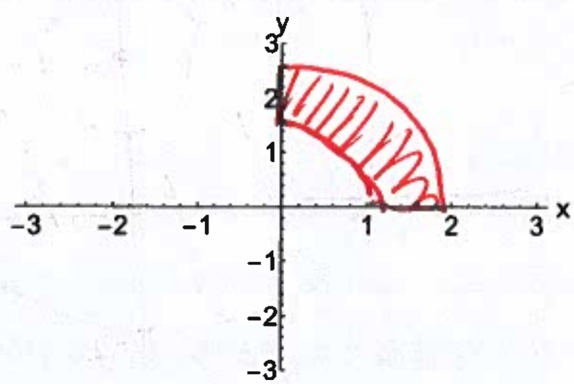
(a) (5 points) $S = \{(X,Y,Z) \in \mathbb{R}^3 : X^2 + Y^2 + Z^2 = 1 \text{ and } Z > \frac{1}{2}\}$





(b) (5 points) $S = \{(X,Y,Z) \in \mathbb{R}^3 : X^2 + Y^2 + Z^2 = 1 \text{ and } 0 \le Z \le \frac{1}{2},\ X \ge 0,\ Y \ge 0\}$





7. (5 points) Write down Euler's formula and show that if $x \in \mathbb{R}$ then

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}.$$

$$e^{ix} = \cos(x) + i\sin(x)$$

$$\Rightarrow \frac{e^{ix} - e^{-ix}}{2i} = \frac{\cos(x) + i\sin(x) - \cos(-x) - i\sin(-x)}{2i}$$

$$= \frac{\cos(x) + i\sin(x) - \cos(x) + i\sin(x)}{2i}$$

$$= \sin(x)$$

8. (10 points) Do either part (a) or part (b). You only need to do one problem to receive full credit. If you attempt both part (a) and part (b) circle or otherwise indicate which problem you want graded. You will not receive extra points if you do both problems.

(a) (5 points) If $z_1, z_2 \in \mathbb{C}$, prove that

$$|z_1 z_2| = |z_1||z_2|.$$

(b) (5 points) If $z \in \mathbb{C}$, prove that

$$z\bar{z} = |z|^2.$$

(a) Let $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$, where $r_1, r_2 \geq 0$ and $\theta_1, \theta_2 \in \mathbb{R}$. Therefore,

$$|z_1 z_2| = |r_1 r_2 e^{i(\theta_1+\theta_2)}| = |r_1 r_2| \cdot |e^{i(\theta_1+\theta_2)}| = r_1 r_2 = |z_1| \cdot |z_2|$$

(b) Let $z = re^{i\theta}$, where $r \geq 0$ and $\theta \in \mathbb{R}$. Therefore,

$$z\bar{z} = re^{i\theta} re^{-i\theta} = r^2 = |z|^2.$$

9. (5 points) **(Undergraduate Problem)** Do either part (a) or part (b). You only need to do one problem to receive full credit. If you attempt both part (a) and part (b) circle or otherwise indicate which problem you want graded. You will not receive extra points if you do both problems.

   (a) (5 points) Prove that for all $\theta_1, \theta_2 \in [0, 2\pi)$,

   $$\cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2).$$

   (b) (5 points) Prove that for all $\theta \in [0, 2\pi)$,

   $$\sin(2\theta) = 2\sin(\theta)\cos(\theta).$$

a) $\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) = e^{i(\theta_1+\theta_2)}$

$= e^{i\theta_1}e^{i\theta_2}$

$= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1$

$\Rightarrow \cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$

b.) $\cos 2\theta + i\sin 2\theta = e^{i2\theta}$

$= (e^{i\theta})^2$

$= (\cos\theta + i\sin\theta)^2$

$= \cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta.$

$\Rightarrow 2\cos\theta\sin\theta = \sin 2\theta.$

10. **(Graduate Problem)**

(a) (5 points) Prove that for all $\theta \in [0, 2\pi)$ and $n \in \mathbb{N}$,

$$\left(\frac{1+i\tan(\theta)}{1-i\tan(\theta)}\right)^n = \frac{1+i\tan(n\theta)}{1-i\tan(n\theta)}.$$

$$\left(\frac{1+i\tan\theta}{1-i\tan\theta}\right)^n = \left(\frac{1+i\frac{\sin\theta}{\cos\theta}}{1-i\frac{\sin\theta}{\cos\theta}}\right)^n$$

$$= \left(\frac{\cos\theta+i\sin\theta}{\cos\theta-i\sin\theta}\right)^n$$

$$= \left(\frac{e^{i\theta}}{e^{-i\theta}}\right)^n$$

$$= \frac{e^{in\theta}}{e^{-in\theta}}$$

$$= \frac{\cos(n\theta)+i\sin(n\theta)}{\cos(n\theta)-i\sin(n\theta)}$$

$$= \frac{1+i\frac{\sin(n\theta)}{\cos(n\theta)}}{1-i\frac{\sin(n\theta)}{\cos(n\theta)}}$$

$$= \frac{1+i\tan(n\theta)}{1-i\tan(n\theta)}.$$

11. (10 points) (**Undergraduate Problem**) Do either part (a) or part (b). You only need to do one problem to receive full credit. If you attempt both part (a) and part (b) circle or otherwise indicate which problem you want graded. You will not receive extra points if you do both problems.

(a) (10 points) Consider the sequence $z_n \in \mathbb{C}$ defined by

$$z_n = \frac{\sin\left(\frac{n\pi}{2}\right) + in}{n}.$$

Prove that

$$\lim_{n\to\infty} z_n = i.$$

(b) (10 points) Suppose that $z_n \in \mathbb{C}$ satisfies $z_n \to 0$ as $n \to \infty$ and $z_n \neq 0$. Prove that

$$\lim_{n\to\infty} z_n e^{\frac{i}{|z_n|}} = 0.$$

(a) $$\left| Z_n - in \right| = \left| \frac{\sin\left(\frac{n\pi}{2}\right) + in}{n} - in \right|$$
$$= \left| \frac{\sin\left(\frac{n\pi}{2}\right)}{n} \right|$$
$$\leq \frac{1}{n}.$$

Therefore, by squeeze theorem $\lim_{n\to\infty} |Z_n - in| = 0$ and thus $\lim_{n\to\infty} Z_n = i$.

(b). $$\left| Z_n e^{i/|Z_n|} - 0 \right| = \left| Z_n e^{i/|Z_n|} \right|$$
$$= |Z_n|$$

Consequently, $\lim_{n\to\infty} |Z_n e^{i/|Z_n|}| = 0$ and thus

$$\lim_{n\to\infty} Z_n e^{i/|Z_n|} = 0.$$

12. **(Graduate Problem)** Do either part (a) or part (b). You only need to do one problem to receive full credit. If you attempt both part (a) and part (b) circle or otherwise indicate which problem you want graded. You will not receive extra points if you do both problems.

(a) (10 points) Suppose that $z_n, w_n \in \mathbb{C}$ satisfy $z_n \to z_0$ and $w_n \to w_0$ as $n \to \infty$. Prove that $z_n w_n \to z_0 w_0$ as $n \to \infty$. Give a specific example to show that the converse is not true.

(b) (10 points) Suppose $z_n \in \mathbb{C}$ satisfies $z_n \to z_0$ as $n \to \infty$. Prove that $|z_n| \to |z_0|$ as $n \to \infty$. Give a specific example to show that the converse is not true.

(a) Since $z_n \to z_0, w_n \to w_0$ we know that

$$\lim_{n\to\infty} |z_n - z_0| = \lim_{n\to\infty} |w_n - w_0| = 0.$$

Moreover,

$$\begin{aligned} |z_n w_n - z_0 w_0| &= |z_n w_n - z_0 w_n + z_0 w_n - z_0 w_0| \\ &= |(z_n - z_0) w_n + z_0 (w_n - w_0)| \\ &\leq |z_n - z_0| \cdot |w_n| + |z_0| \cdot |w_n - w_0| \\ &\leq M \cdot |z_n - z_0| + |z_0| |w_n - w_0|, \end{aligned}$$

where $M = \sup_{n\in\mathbb{N}} |w_n| < \infty$. Consequently, by squeeze theorem we have that

$$\lim_{n\to\infty} |z_n w_n - z_0 w_0| = 0$$

and thus $\lim_{n\to\infty} z_n w_n = z_0 w_0$.

(b) Since $z_n \to z_0$ it follows that $\lim_{n\to\infty} |z_n - z_0| = 0$. Furthermore, by the reverse triangle inequality

$$\big| |z_n| - |z_0| \big| \leq |z_n - z_0|$$

and thus by the squeeze theorem

$$\lim_{n\to\infty} \big| |z_n| - |z_0| \big| = 0.$$

Therefore,

$$\lim_{n\to\infty} |z_n| = |z_0|.$$