MTH 317/617
Fall 2022
Exam 2
10/28/22

Name (Print): Key

This exam contains 10 pages (including this cover page) and 10 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You are required to show your work on each problem on this exam. The following rules apply:

- **Resources:** You are allowed to use your text, notes, and a calculator. You are not allowed to use the internet, software or any other external resource.
- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Undergraduate Problems:** Questions labeled as **"Undergraduate problems"** will only count for undergraduate students. Graduate students do not have to complete these problems.
- **Graduate Problems:** Questions labeled as **"Graduate problems"** must be completed by the graduate students to receive credit. Undergraduate students can complete these problems for extra credit.
- **Short answer questions:** Questions labeled as **"Short Answer"** can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit.** An incorrect answer supported by substantially correct calculations and explanations might receive partial credit.

Problem	Points	Score
1	15	
2	10	
3	15	
4	15	
5	10	
6	0	
7	10	
8	15	
9	10	
10	0	
Total:	100	

Do not write in the table to the right.

1. (15 points) (**Short Answer**) Determine if the following statement is correct (**C**) or incorrect (**I**). Just circle **C** or **I**. No need to show any work. In order for a statement to be correct it must be true in all cases.

   C **I** If $z_1, z_2 \in \mathbb{C}$ then $\mathrm{Log}(z_1 z_2) = \mathrm{Log}(z_1) + \mathrm{Log}(z_2)$.

   C **I** $z \in \mathbb{C}$ then $\log(e^z) = z$.

   C **I** If $z \in \mathbb{C}$ then $e^{\log(z)} = z$.

   C **I** If $z \in \mathbb{C}$ then $|\sin(z)| \leq 1$.

   C **I** If $f, g : \mathbb{C} \mapsto \mathbb{C}$ are analytic functions then then the product of these two functions is analytic.

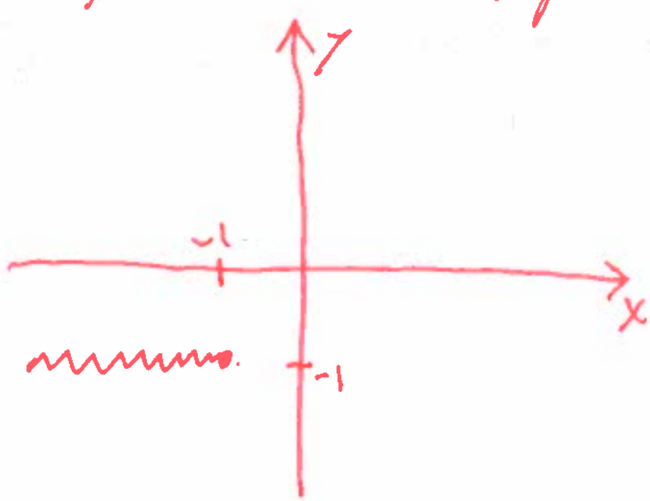
2. (10 points) Determine the domain of analyticity for the function $f(z) = \mathrm{Log}(z + i + 1)$ and sketch the branch cuts on the complex plane.

If we let $z = x + iy$ then f is not analytic when

$x + 1 \leq 0$ and $1 + y = 0$

$\Rightarrow x \leq -1$ and $y = -1$.

Therefore, the branch cut is given by:



3. (15 points) Consider the following rational function

$$R(z) = \frac{z+1}{(z^2-1)(z+i)}$$

defined on $\mathbb{C}$ except for points in which $R(z)$ is not continuous.

(a) (2 points) Determine the zeros of $R(z)$ and their multiplicities.

$R(z) = \dfrac{1}{(z-1)(z+i)}$ and thus has no zeros.

(b) (3 points) Determine the poles of $R(z)$ and their multiplicities.

poles at $z=1$ and $z=-i$ each with multiplicity 1.

(c) (10 points) Recall that if $\zeta \in \mathbb{C}$ is a pole of $R(z)$ then the coefficient of of $1/(z-\zeta)$ in the partial fraction decomposition of $R(z)$ is called the **residue** of $R(z)$ at ζ. Compute the residue for each of the poles you found in part (b).

$$\frac{1}{(z-1)(z+i)} = \frac{A}{z-1} + \frac{B}{z+i} = \frac{A(z+i)+B(z-1)}{(z-1)(z+i)}$$

$z=1$:
$1 = A(1+i)$
$A = \dfrac{1}{1+i}$

$z=-i$:
$1 = B(-i-1)$
$B = \dfrac{-1}{1+i}$

Therefore,

$Res(z=1) = \dfrac{1}{1+i}$

$Res(z=-i) = \dfrac{-1}{1+i}$

4. (15 points) Express all of the following in the standard form $x + iy$ where $x, y \in \mathbb{R}$. In each problem assume the principal branch for Log and assume complex powers also use the principal branch.

(a) (5 points) $\operatorname{Log}(1+i)$

$\operatorname{Log}(1+i) = \ln(\sqrt{2}) + i\pi/4.$

(b) (5 points) $\operatorname{Log}\left(\frac{1}{2}e^{\frac{3}{2}\pi i}\right)$

$\operatorname{Log}\left(\frac{1}{2}e^{3/2\pi i}\right) = \ln(1/2) - i\pi/2.$

(c) (5 points) i^{1+i}

$$i^{1+i} = e^{(1+i)\operatorname{Log}(i)}$$
$$= e^{(1+i)(\pi i/2)}$$
$$= e^{-\pi/2} e^{i\pi/2}$$
$$= i e^{-\pi/2}.$$

5. (10 points) **(Undergraduate Problem:)** Show that the function

$$f(x+iy) = e^x(\cos(y) + i\sin(y))$$

is analytic on all of $\mathbb{C}$, and find its derivative.

$$u(x,y) = e^x\cos(y)$$
$$v(x,y) = e^x\sin(y)$$

$$\frac{\partial u}{\partial x} = e^x\cos(y), \quad \frac{\partial u}{\partial y} = -e^x\sin(y)$$
$$\frac{\partial v}{\partial y} = e^x\cos(y), \quad \frac{\partial v}{\partial x} = e^x\sin(y)$$

and thus the Cauchy Riemann equations are satisfied.
Furthermore,

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$= e^x\cos(y) + i e^x\sin(y).$$

6. (0 points) **(Graduate Problem:)** Suppose $f : \mathbb{C} \mapsto \mathbb{C}$ defined by $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ is analytic in a domain D. We define the partial derivatives of f by

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \text{ and } \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}.$$

Prove that the Cauchy Riemann equations are equivalent to

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0.$$

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0 \Leftrightarrow \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = 0$$

$$\Leftrightarrow \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = 0$$

$$\Leftrightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y},$$

7. (10 points) Suppose that the function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, where $u, v : D \mapsto \mathbb{R}$. In the following problems you can assume that harmonic functions have continuous partial derivatives of all orders.

(a) (5 points) Show that $\frac{\partial u}{\partial x}$ is harmonic in D.

$$\frac{\partial^2}{\partial x^2}\frac{\partial u}{\partial x} + \frac{\partial^2}{\partial y^2}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$$

(b) (5 points) Determine the harmonic conjugate of $\frac{\partial u}{\partial x}$.

Since

$$\frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x}$$

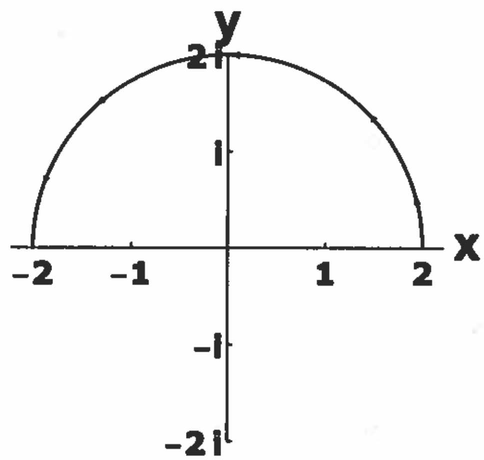
and

$$\frac{\partial}{\partial y}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left(-\frac{\partial v}{\partial x}\right) = -\frac{\partial}{\partial x}\frac{\partial v}{\partial x}$$

and thus the harmonic conjugate is given by

$$\frac{\partial v}{\partial x}.$$

8. (15 points) Let C be the semicircle connecting $z = 2$ to $z = -2$ oriented in the positive sense; see the below figure.



(a) (5 points) Write down a parametrization of this contour. Be sure to include your domain of parametrization.

$z(t) = 2e^{it}, \quad t \in [0, \pi]$

(b) (5 points) Compute the following contour integral: $\int_C \sin(z)dz$.

$\int_C \sin(z)dz = -\cos(z)\Big|_2^{-2} = -\cos(-2) + \cos(2) = 0$

(c) (5 points) Compute the following contour integral: $\int_C |z|dz$.

$\int_C |z|dt = \int_0^\pi |2e^{it}| 2ie^{it}dt$

$= \int_0^\pi 4ie^{it}dt$

$= 4e^{it}\Big|_0^\pi$

$= 4e^{i\pi} - 4$

$= -8$

9. (10 points) **(Undergaduate Problem:)** Let $n \in \mathbb{Z}$ and let $C_R(z_0)$ denote the positively oriented circle with center at z_0 with radius $R > 0$. By explicitly parametrizing $C_R(z_0)$ and computing the integral, show that

$$\int_{C_R(z_0)} (\overline{z - z_0})^n dz = \begin{cases} 0 & \text{if } n \neq 1 \\ 2\pi i R^2 & \text{if } n = 1 \end{cases}.$$

$z(t) = Re^{it} + z_0, \quad t \in [0, 2\pi].$

$$\int_{C_R(z_0)} (\overline{z - z_0})^n dz = \int_0^{2\pi} R^n e^{-int} iRe^{it} dt$$

$$= \int_0^{2\pi} R^{n+1} e^{i(1-n)t} dt$$

$$= \begin{cases} \frac{R^{n+1} i}{i(1-n)} e^{i(1-n)t} \Big|_0^{2\pi}, & n \neq 1 \\ 2\pi i R^2, & n = 1 \end{cases}$$

$$= \begin{cases} 0, & n \neq 1 \\ 2\pi i R^2, & n = 1 \end{cases}$$

10. (0 points) **(Graduate Problem:)** Let f be an analytic function with a continuous derivative satisfying $|f'(z)| \leq M$ for all z in the disk $D = \{z \in \mathbb{C} : |z| < 1\}$. Show that

$$|f(z_2) - f(z_1)| \leq M|z_2 - z_1|.$$

Hint: Think about integrating $f'(z)$ along a line segment connecting z_1 to z_2.

$$|f(z_2)-f(z_1)| = \left|\int_{z_1}^{z_2} f'(z)\,dz\right|$$

$$\leq \int_{z_1}^{z_2} |f'(z)||dz|$$

$$\leq \int_{z_1}^{z_2} M|dz|$$

$$= M|z_2 - z_1|$$