MTH 317/617
Homework #10

Due Date: November 18, 2022

1 Problems for Everyone

1. Find the Laurent series for the function $f(z) = \dfrac{1}{z + z^2}$ in each of the following domains.

   (a) $0 < |z| < 1$

   (b) $|z| > 1$

   (c) $0 < |z + 1| < 1$

   (d) $1 < |z + 1|$

2 2. Find the Laurent series for the following functions in the indicated domains

   (a) $f(z) = \dfrac{\sin(2z)}{z^3}$ in $|z| > 0$

   (b) $f(z) = \dfrac{(z+1)}{z(z-4)}$ in $0 < |z - 4| < 4$

   (c) $f(z) = z^2 \cos\left(\dfrac{1}{3z}\right)$ in $|z| > 0$

   (d) $f(z) = \dfrac{e^{1/z}}{z^2 - 1}$ in $|z| > 1$.

2 3. Recall, that for a function $f(z)$ with an isolated singularity located at $z = z_0$, the residue at z_0 is the coefficient of $1/(z - z_0)$ in the Laurent series of $f(z)$ about z_0. Determine all of the isolated singularities of each of the following functions and compute the residue at each singularity.

   (a) $\dfrac{1+z}{z}$

   (b) $\dfrac{1+z}{z^2 + 2z + 2}$

   (c) $\dfrac{1 + e^z}{z^2} + \dfrac{2}{z}$

   (d) $\dfrac{\sin(z^2)}{z^2(z^2+1)}$

   (e) $\dfrac{1 - \cos(z)}{z^2}$

   (f) $\dfrac{1}{z \sin(z)}$

(g) $\exp(z + z^{-1})$

(h) $\dfrac{\cot(\pi z)}{z+1}$

2 4. Evaluate the following contour integrals

(a) $\displaystyle\int_{|z|=1} \frac{z^2+3z-1}{z(z^2-3)}dz$

(b) $\displaystyle\int_{|z|=1} \frac{\sin(z)}{z^6}dz$

(c) $\displaystyle\int_{|z|=4} z\tan(z)dz$

(d) $\displaystyle\int_{|z|=1} \frac{e^{z^2}}{z^6}dz$

(e) $\displaystyle\int_{|z|=1} z^4\left(e^{z^{-1}} + z^2\right)dz$

(f) $\displaystyle\int_{|z|=1} \cos\left(\frac{1}{z^2}\right)e^{z^{-1}}dz$

(g) $\displaystyle\int_{|z|=1} \frac{1}{z^2(e^z-1)}dz$

2 5. Verify each of the following integrals

(a) $\displaystyle\int_0^{2\pi} \frac{1}{2+\sin(\theta)}d\theta = \frac{2\pi}{\sqrt{3}}$

(b) $\displaystyle\int_0^{2\pi} \frac{8}{5+2\cos(\theta)}d\theta = \frac{8\pi}{\sqrt{21}}$

(c) $\displaystyle\int_{-\pi}^{\pi} \frac{1}{1+\sin^2(\theta)}d\theta = \pi\sqrt{2}$

2 6. Verify each of the following integrals. You must show that the contribution along any circular arc in the complex plane yields 0 in the limit as the radius of the arc goes to infinity.

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^4}dx = \frac{\pi}{\sqrt{2}}$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)}dx = \frac{\pi}{6}$

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^4+x^2+1}dx = \frac{\pi}{\sqrt{3}}$

<u>Homework #10</u>

<u>#2.</u>

Find the Laurent series for the following functions in the indicated domain.

(a) $f(z) = \dfrac{\sin(2z)}{z^3}$, $|z| > 0$

(b) $f(z) = \dfrac{z+1}{z(z-4)}$, $0 < |z-4| < 4$

(c) $f(z) = z^2 \cos\left(\dfrac{1}{3z}\right)$, $|z| > 0$

(d) $f(z) = \dfrac{e^{1/z}}{z^2-1}$, $|z| > 1$.

<u>Solution:</u>

(a) $$\frac{\sin(2z)}{z^3} = \frac{2z - 8z^3/3! + 32z^5/5! + \dots}{z^3} = \frac{2}{z^2} - \frac{8}{3!} + \frac{32z^2}{5!} + \dots$$

(b) Letting $w = z-4$ it follows that $0 < |w| < 4$ and that

$$\frac{z+1}{z(z-4)} = \frac{w+5}{(w+4)w} = \frac{w+5}{4w(1+w/4)} = \frac{w+5}{4w}\left(1 - \frac{w}{4} + \frac{w^2}{16} + \dots\right)$$

$$\therefore \frac{z+1}{z(z-4)} = \frac{5}{4w} + \frac{1}{4}\left(5 - \frac{5}{4}\right) + \frac{1}{4}\left(\frac{5}{16} - \frac{1}{4}\right)w + \dots$$

$$= \frac{5}{4(z-4)} \cancel{+} \frac{\cancel{1}5}{16} + \frac{1}{16}(z-4) + \dots$$

(c). $$z^2\cos\left(\frac{1}{3z}\right) = z^2\left(1 - \frac{1}{(9z^2)2!} + \frac{1}{81z^4 \cdot 4!} + \dots\right)$$

$$= z^2 - \frac{1}{18} + \frac{1}{4!\,81 z^2} + \dots$$

(d) $\dfrac{e^{1/z}}{z^2-1} = \dfrac{e^{1/z}}{z^2(1-1/z^2)} = \dfrac{(1+1/z+1/2!z^2+\cdots)(1+1/z^2+1/z^4+\cdots)}{z^2}$

$$\Rightarrow \frac{e^{1/z}}{z^2-1} = \frac{1+1/z+(1+1/2)1/z^2+\ldots}{z^2}$$

$$= \frac{1}{z^2} + \frac{1}{z^3} + \frac{3}{2}\cdot\frac{1}{z^4} + \ldots$$

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#3

Determine all of the isolated singularities of each of the following functions and compute the residue at each singularity.

(b) $\dfrac{1+z}{z^2+2z+2}$

(c) $\dfrac{1+e^z}{z^2} + \dfrac{2}{z}$

(d) $\dfrac{\sin(z^2)}{z^2(z^2+1)}$

(f) $\dfrac{1}{z\sin(z)}$

Solution:

(b) The singularities are at

$$z = -1 \pm i$$

and thus

$$\frac{1+z}{(z+1-i)(z+1+i)}.$$

Consequently, $\text{res}(f(z); -1\pm i) = \frac{1}{2}$

$$\text{Res}(f(z); 1+i) = \left.\frac{1+z}{z-1+i}\right|_{1+i} = \frac{2+i}{2i}, \quad \text{Res}(f(z); 1-i) = \left.\frac{1+z}{z-1-i}\right|_{1-i} = \frac{2-i}{-2i}.$$

(c) The only singularity is at $z=0$ and

$$\frac{1+e^z}{z^2}+\frac{2}{z}=\frac{1}{z^2}+\frac{1+z+\ldots}{z^2}+\frac{2}{z}=\frac{3}{z^2}+\frac{3}{z}+\ldots$$

Therefore,

$$\mathrm{Res}(f(z);0)=3.$$

(d) The singularities are at $z=0,\pm i$.

$z=0$:

$$\frac{\sin(z^2)}{z^2(z^2+1)}=\frac{z^2-\frac{1}{3!}z^6+\ldots}{z^2(z^2+1)}=\frac{1-\frac{1}{3!}z^4+\ldots}{1+z^2}$$

and consequently $\mathrm{Res}(f(z);0)=0$.

$z=\pm i$:

$$\frac{\sin(z^2)}{z^2(1+z^2)}=\frac{\sin(z^2)}{z^2(z-i)(z+i)}$$

and thus

$$\mathrm{Res}(f(z);i)=\frac{\sin(-1)}{-2i}=\frac{-\sin(1)}{2}i.$$

$$\mathrm{Res}(f(z);-i)=\frac{\sin(-1)}{-i\cdot(-2i)}=\frac{\sin(1)}{2}i.$$

(f) The singularities are at $z=0, n\pi$.

$z=0$:

$$\frac{1}{z\sin(z)}=\frac{1}{z(z-z^3/3!+\ldots)}=\frac{1}{z^2(1-z^2/3!+\ldots)}=\frac{1}{z^2}\left(1+g(z)+g(z)^2+\ldots\right),$$

where $g(z)=z^2/3!+\ldots$. Since to lowest order $g(z)$ is quadratic it follows that

$$\mathrm{Res}(f(z);0)=0.$$

$z = n\pi$:

Taylor expanding we have that

$$\sin(z) = \sin(n\pi) + \cos(n\pi)(z-n\pi) + c(z-n\pi)^2 + \dots$$
$$= (-1)^n (z-n\pi) + c(z-n\pi)^2 + \dots,$$

where c is some constant. Therefore,

$$\frac{1}{z\sin(z)} = \frac{1}{z(-1)^n(z-n\pi)(1+d(z-n\pi)+\dots)},$$

where d is some constant. Since near $z=n\pi$ we have that

$$g(z) = \frac{1}{(-1)^n z(1+d(z-n\pi)+\dots)}$$

is analytic it follows that

$$\mathrm{Res}(f(z); n\pi) = g(n\pi) = \frac{(-1)^n}{n\pi}.$$

$\blacksquare$

#4.

Evaluate the following contour integrals.

(a) $\displaystyle\int_{|z|=1} \frac{z^2+3z-1}{z(z^2-3)}\,dz$

(b) $\displaystyle\int_{|z|=1} \frac{\sin(z)}{z^6}\,dz$

(c) $\displaystyle\int_{|z|=1} z\tan(z)\,dz$

(g) $\displaystyle\int_{|z|=1} \frac{1}{z^2(e^z-1)}\,dz$

Solution:

(a) $$\int_{|z|=1} \frac{z^2+3z-1}{z(z^2-3)}\,dz = 2\pi i\,\mathrm{Res}(f;0) = 2\pi i\left(\frac{z^2+3z-1}{z^2-3}\right)\Big|_0 = \frac{2\pi i}{3}.$$

(b) $$\int_{|z|=1} \frac{\sin(z)}{z^6}\,dz = \int_{|z|=1} \frac{z - z^3/3! + z^5/5! + \dots}{z^6}\,dz = \int_{|z|=1} \frac{1}{5!\,z}\,dz = \frac{2\pi i}{5!}$$

(c) $$\int_{|z|=4} z\tan(z)\,dz = 2\pi i\left(\mathrm{Res}(f;\pi/2) + \mathrm{Res}(f;-\pi/2)\right)$$

$z=\pi/2$:

$$z\tan z = \frac{z\sin(z)}{\cos(z)} = \frac{z\sin(z)}{-(z-\pi/2)+c(z-\pi/2)^3+\dots} = \frac{z\sin(z)}{-(z-\pi/2)(1-c(z-\pi/2)^2+\dots)}$$

Therefore,

$$\mathrm{Res}(f;\pi/2) = -\pi/2\,\sin(\pi/2) = -\pi/2.$$

$z=-\pi/2$:

$$z\tan(z) = \frac{z\sin(z)}{(z+\pi/2)+c(z+\pi/2)^3+\dots} = \frac{z\sin(z)}{(z+\pi/2)(1+c(z+\pi/2)^2+\dots)}.$$

Therefore,

$$\mathrm{Res}(f;-\pi/2) = -\pi/2\,\sin(-\pi/2) = \pi/2.$$

Consequently,

$$\int_{|z|=4} z\tan(z)\,dz = 2\pi i\left(\frac{-\pi}{2}+\frac{\pi}{2}\right) = 0.$$

(g) $$\int_{|z|=1} \frac{1}{z^2(e^z-1)}\,dz = \int_{|z|=1} \frac{1}{z^2(z+z^2/2!+\dots)}\,dz = \int_{|z|=1} \frac{1}{z^3(1+z/2!+z^2/3!+z^3/4!+\dots)}\,dz$$

$$\Rightarrow \int_{|z|=1} \frac{1}{z^2(e^z-1)}\,dz = \int_{|z|=1} \frac{1}{z^3}\left(1-\left(\frac{z}{2!}+\frac{z^2}{3!}+\dots\right)+\left(\frac{z}{2!}+\frac{z^2}{3!}+\dots\right)^2+\dots\right)dz$$

$$\Rightarrow \int_{|z|=1} \frac{1}{z^2(e^z-1)}\,dz = \int_{|z|=1} \frac{1}{z^3}\left(\frac{-1}{3!}+\frac{1}{(2!)^2}\right)z^2\,dz = 2\pi i\left(\frac{1}{4}-\frac{1}{6}\right).$$

Therefore,

$$\int_{|z|=1} \frac{1}{z^2(e^z-1)}\,dz = \frac{2\pi i}{12} = \frac{\pi i}{6}.$$

$\blacksquare$

#5

Compute the following integrals

(a) $\displaystyle\int_0^{2\pi} \frac{1}{2+\sin\theta}\,d\theta$

(b) $\displaystyle\int_0^{2\pi} \frac{8}{5+2\cos\theta}\,d\theta$

Solution:

(a) Letting $z=e^{i\theta}$ it follows that $dz=ie^{i\theta}d\theta$ and $\sin\theta=\frac{1}{2i}(z-z^{-1})$. Therefore

$$\int_0^{2\pi} \frac{1}{2+\sin\theta}\,d\theta = \int_{|z|=1} \frac{1}{iz(2+\frac{1}{2i}(z-z^{-1}))}\,dz$$

$$= \int_{|z|=1} \frac{2}{(4iz+z^2-1)}\,dz$$

$$= \int_{|z|=1} f(z)\,dz.$$

The function f has singularities at

$$z=-2i\pm\sqrt{3}i.$$

Therefore,

$$\int_0^{2\pi} \frac{1}{2+\sin\theta}\,d\theta = 2\pi i \,\mathrm{Res}(f; -2i+\sqrt{3}i)$$

$$= 2\pi i \left(\frac{2}{z+2i+\sqrt{3}i} \right)\Bigg|_{-2i+\sqrt{3}i}$$

$$= 2\pi i \left(\frac{2}{2\sqrt{3}i} \right)$$

$$= \frac{2\pi}{\sqrt{3}}.$$

(b) Letting $z = e^{i\theta}$ it follows that $dz = ie^{i\theta}d\theta$ and $\cos(\theta) = \frac{1}{2}(z+z^{-1})$.

Therefore,

$$\int_0^{2\pi} \frac{8}{5+2\cos\theta}\,d\theta = \int_{|z|=1} \frac{8}{iz(5+z+z^{-1})}\,dz$$

$$= \int_{|z|=1} \frac{-8i}{(z^2+5z+1)}\,dz$$

$$= \int_{|z|=1} f(z)\,dz.$$

The function f has singularities at

$$z = -5/2 \pm 1/2\sqrt{21}$$

Therefore,

$$\int_0^{2\pi} \frac{8}{5+2\cos\theta}\,d\theta = 2\pi i \,\mathrm{Res}(f; -5/2 + \sqrt{21}/2)$$

$$= 2\pi i \left(\frac{-8i}{z+5/2+\sqrt{21}/2} \right)\Bigg|_{-5/2+\sqrt{21}/2}$$

$$= 2\pi i \left(\frac{-8i}{\sqrt{21}} \right)$$

$$= \frac{16\pi}{\sqrt{21}}$$

#6.

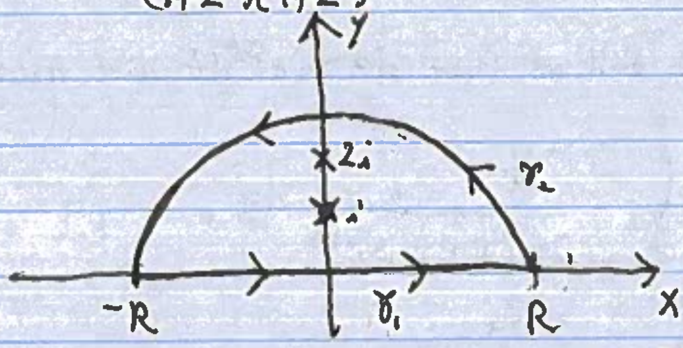
Compute the following integrals.

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(1+x^2)(4+x^2)}\,dx$

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^4+x^2+1}\,dx$

Solution:

(b) Let $f(z) = \dfrac{1}{(1+z^2)(4+z^2)}$ and consider the contour Γ drawn below.



Therefore, along γ_2 we have $z = Re^{i\theta}$ and thus $dz = Rie^{i\theta}$. Consequently,

$$\left|\int_{\gamma_2} f(z)\,dz\right| \le \int_0^{\pi} \frac{R}{|(1+R^2e^{2i\theta})(4+R^2e^{2i\theta})|}\,d\theta$$

$$\le \int_0^{\pi} \frac{R}{(R^2-1)(R^2-4)}\,d\theta$$

$$= \frac{\pi R}{(R^2-1)(R^2-4)}.$$

and thus by the squeeze theorem

$$\lim_{R\to\infty} \int_{\gamma_2} f(z)\,dz = 0. \quad (*)$$

Calculating the residues we have:

$$\operatorname{Res}(f; i) = \frac{1}{(z+i)(4+z^2)}\bigg|_{z=i} = \frac{1}{2i\cdot 3} = \frac{1}{6i}.$$

$$\operatorname{Res}(f; 2i) = \frac{1}{(z+2i)(1+z^2)}\bigg|_{z=2i} = \frac{1}{4i}\frac{1}{(-3)} = -\frac{1}{12i}.$$

Consequently,

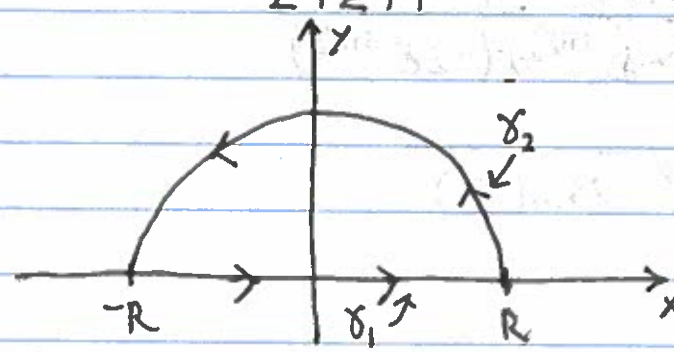
$$\int_\Gamma f(z)\,dz = 2\pi i\left(\frac{1}{6i} - \frac{1}{12i}\right) = \frac{\pi}{6}. \quad (**)$$

Finally, by $(*)$ and $(**)$ we have

$$\frac{\pi}{6} = \lim_{R\to\infty}\int_\Gamma f(z)\,dz = \lim_{R\to\infty}\int_{\gamma_1} f(z)\,dz + \lim_{R\to\infty}\int_{\gamma_2} f(z)\,dz$$

$$\Rightarrow \frac{\pi}{6} = \int_{-\infty}^{\infty} \frac{1}{(1+z^2)(4+z^2)}\,dz.$$

(c). Let $f(z) = \dfrac{1}{z^4+z^2+1}$ and consider the contour draw below



Therefore, along γ_2 we have $z = Re^{i\theta}$ and thus $dz = Rie^{i\theta}d\theta$.

Consequently,

$$\left|\int_{\gamma_2} f(z)\,dz\right| \le \int_0^\pi \frac{R}{|R^4+R^2+1|}\,d\theta \le \int_0^\pi \frac{R}{R^4-R^2-1}\,d\theta = \frac{\pi R}{R^4-R^2-1}$$

and thus by squeeze theorem

$$\lim_{R\to\infty}\int_{\gamma_2} f(z)\,dz = 0.$$

Since $z^4+z^2+1=\left(z^2+\frac{1-\sqrt{3}i}{2}\right)\left(z^2+\frac{1+\sqrt{3}i}{2}\right)$

$$=(z^2+e^{i\pi/3})(z^2+e^{-i\pi/3})$$
$$=(z+e^{i\pi/6})(z-e^{i\pi/6})(z+e^{-i\pi/6})(z-e^{-i\pi/6})$$

It follows that the singularities within the contour are $z=e^{i\pi/6}$ and $z=-e^{-i\pi/6}$

The residues are thus given by

$$\underline{\mathrm{Res}(f,e^{i\pi/6})}=\frac{1}{2e^{i\pi/6}(e^{i\pi/6}+e^{-i\pi/6})(e^{i\pi/6}-e^{-i\pi/6})}$$
$$=\frac{1}{2e^{i\pi/6}(e^{i\pi/3}-e^{-i\pi/3})}$$
$$=\frac{1}{2e^{i\pi/6}\cdot 2i\sin(\pi/3)}$$

$$\underline{\mathrm{Res}(f,-e^{-i\pi/6})}=\frac{1}{(e^{i\pi/6}-e^{-i\pi/6})(-e^{-i\pi/6}-e^{i\pi/6})(-2e^{-i\pi/6})}$$
$$=-\frac{1}{(e^{i\pi/3}-e^{-i\pi/3})(-2e^{-i\pi/6})}$$
$$=\frac{-1}{2e^{-i\pi/6}2i\sin(\pi/3)}$$

Therefore,

$$\int_\Gamma f(z)\,dz=\frac{2\pi i}{4i\sin(\pi/3)}\left(\frac{1}{e^{i\pi/6}}-\frac{1}{e^{-i\pi/6}}\right)=\frac{2\pi i}{4i\sin(\pi/3)}(e^{-i\pi/6}-e^{i\pi/6})$$

$$\Rightarrow\int_\Gamma f(z)\,dz=\frac{\pi}{\sin(\pi/3)}\sin(\pi/6)$$
$$=\frac{\pi}{\sqrt{3}}.$$