MTH 317/617
Homework #11

Due Date: December 02, 2022

1 Problems for Everyone

1. Verify each of the following.

   (a) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 e^x}{e^{2x}+1}\,dx = \frac{\pi^3}{6}$

   (b) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{3e^x + e^{-x}}\,dx = \frac{\pi}{2\sqrt{3}}$

   (c) $\displaystyle\int_{-\infty}^{\infty} \frac{x e^x}{e^{2x}+1}\,dx = 0$

2. Verify each of the following

   (a) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(4x)}{1+x^2}\,dx = \pi e^{-4}$

   (b) $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin(3x)}{x^2+2}\,dx = e^{-3\sqrt{2}}\pi$

   (c) $\displaystyle\int_{-\infty}^{\infty} \frac{x\cos(\pi x)}{x^2+x+9}\,dx = \pi e^{-\frac{\sqrt{35}}{2}\pi}$

3. Verify each of the following

   (a) P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{2x\sin(x)}{x^2-a^2}\,dx = 2\pi\cos(a)$, $a \in \mathbb{R}$

   (b) P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(ax)}{x-b}\,dx = \pi\cos(ab)$, $a, b \in \mathbb{R}$

   (c) P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(x)}{x(x^2+1)}\,dx = \pi\left(1-\frac{1}{e}\right)$

4. Verify each of the following

   (a) $\displaystyle\int_{-\infty}^{\infty} \frac{\sqrt{x}}{1+x}\,dx = \pi$

   (b) $\displaystyle\int_{-\infty}^{\infty} \frac{x^\alpha}{x^2-1}\,dx = \frac{\pi}{2\sin(\pi\alpha)}\left[1-\cos(\pi\alpha)\right]$, $\alpha \in \mathbb{R}$, $-1 < \alpha < 1$, $\alpha \neq 0$

Homework #11

#1

Compute the following

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{3e^{x}+e^{-x}}\,dx.$

Solution:

Let $f(z) = \dfrac{1}{3e^{z}+e^{-z}} = \dfrac{e^{z}}{3e^{2z}+1}$ which has a singularity when $3e^{2z}+1=0$ which implies

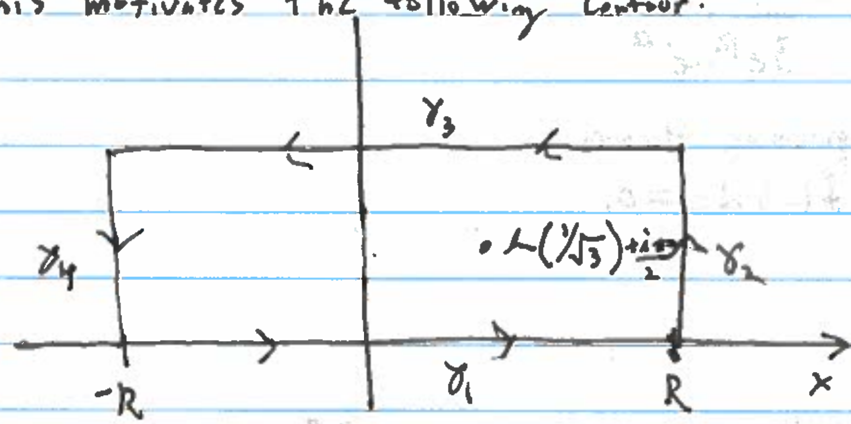
$$e^{2z} = -\tfrac{1}{3} = \tfrac{1}{3}e^{i\pi+2in\pi}$$

$$\Rightarrow 2z = \ln(\tfrac{1}{3}) + i\pi + 2in\pi$$

$$\Rightarrow z = \frac{\ln(\tfrac{1}{3})}{2} + \frac{i\pi}{2} + in\pi.$$

$$= \ln(\tfrac{1}{\sqrt{3}}) + \frac{i\pi}{2} + in\pi.$$

This motivates the following contour:



Taylor expanding about $z^{*} = \ln(\tfrac{1}{\sqrt{3}}) + i\pi/2$ we have

$$3e^{2z}+1 = 6e^{2z}\Big|_{z^*}(z-z^*) + \frac{12e^{2z}}{2!}\Big|_{z^*}(z-z^*)^2 + \dots$$

$$= 6e^{\ln(1/3)+i\pi}(z-z^*) + c(z-z^*)^2 + \dots$$

$$= -2(z-z^*) + c(z-z^*)^2 + \dots$$

and thus

$$\frac{e^{z}}{3e^{2z}+1} = \frac{e^{z}}{\dots}.$$

Consequently,

$$\operatorname{Res}(f; z^*) = \frac{e^{z^*}}{-2} = \frac{e^{\ln(1/\sqrt{3})} e^{i\pi/2}}{-2} = \frac{-i}{2\sqrt{3}}.$$

We now compute the contribution of each integral.

$\underline{\gamma_1}$:

$$\int_{\gamma_1} f(z)\,dz = \int_{-R}^{R} \frac{1}{3e^x + e^{-x}}\,dx$$

$\underline{\gamma_2}$:

$$\left| \int_{\gamma_2} f(z)\,dz \right| = \left| \int_0^{\pi} \frac{i}{3e^{R+iy} + e^{-R-iy}}\,dy \right|$$

$$\leq \int_0^{\pi} \frac{1}{3e^R - e^{-R}}\,dy$$

$$= \frac{\pi}{3e^R - e^{-R}}.$$

Therefore, by squeeze theorem

$$\lim_{R\to\infty} \int_{\gamma_2} f(z)\,dz = 0.$$

$\underline{\gamma_3}$:

$$\int_{\gamma_3} f(z)\,dz = \int_{R}^{-R} \frac{1}{3e^{x+i\pi} + e^{-x-i\pi}}\,dx = \int_{-R}^{R} \frac{1}{3e^x + e^{-x}}\,dx.$$

$\underline{\gamma_4}$:

By similar arguments used to compute $\int_{\gamma_2} f(z)\,dz$, we have

$$\lim_{R\to\infty} \int_{\gamma_4} f(z)\,dz \Rightarrow 0$$

Putting it all together we have that

$$2\pi i\left(\frac{-i}{2\sqrt{3}}\right)=\frac{\pi}{\sqrt{3}}=\lim_{R\to\infty}\int_{\Gamma} f(z)\,dz = 2\int_{-\infty}^{\infty}\frac{1}{3e^{x}+e^{-x}}\,dx.$$

Therefore,

$$\int_{-\infty}^{\infty}\frac{1}{3e^{x}+e^{-x}}\,dx=\frac{\pi}{2\sqrt{3}}.$$

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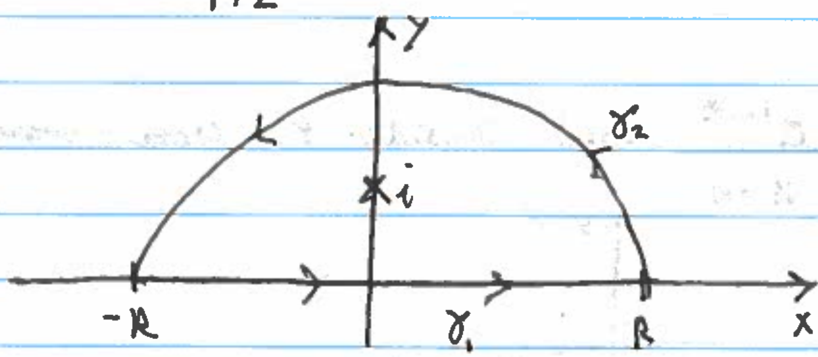
#2.

Compute the following

(a) $\displaystyle\int_{-\infty}^{\infty}\frac{\cos(4x)}{1+x^2}\,dx.$

__Solution:__

Let $f(z)=\dfrac{e^{4iz}}{1+z^2}$ and consider the contour below:



The residue at $z=i$ is given by

$$\operatorname{Res}(f,i)=\frac{e^{-4}}{2i}$$

Computing the contour integrals we have:

__γ_1:__ $$\int_{\gamma_1} f(z)\,dz=\int_{-R}^{R}\frac{\cos(4x)+i\sin(4x)}{1+x^2}\,dx$$

__γ_2:__ $$\left|\int_0^{\pi}\frac{e^{4iRe^{i\theta}}}{1+(4Re^{i\theta})^2}Rie^{i\theta}\,d\theta\right|\le\int_0^{\pi}\frac{e^{-4R\sin\theta}}{R^2-1}R\,d\theta\le\frac{C}{R^2-1}$$

and thus by squeeze theorem $\displaystyle\lim_{R\to\infty}\int_{\gamma_2} f(z)\,dz.$

Putting it all together we have

$$2\pi i \frac{e^{-4}}{2i} = \pi e^{-4} = \lim_{R\to\infty} \int_{\gamma} f(z)\,dz = \int_{-\infty}^{\infty} \frac{\cos(4x)}{1+x^2}\,dx + i\int_{-\infty}^{\infty} \frac{\sin(4x)}{1+x^2}\,dx$$

and thus

$$\int_{-\infty}^{\infty} \frac{\cos(4x)}{1+x^2}\,dx = \pi e^{-4}.$$

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#3.

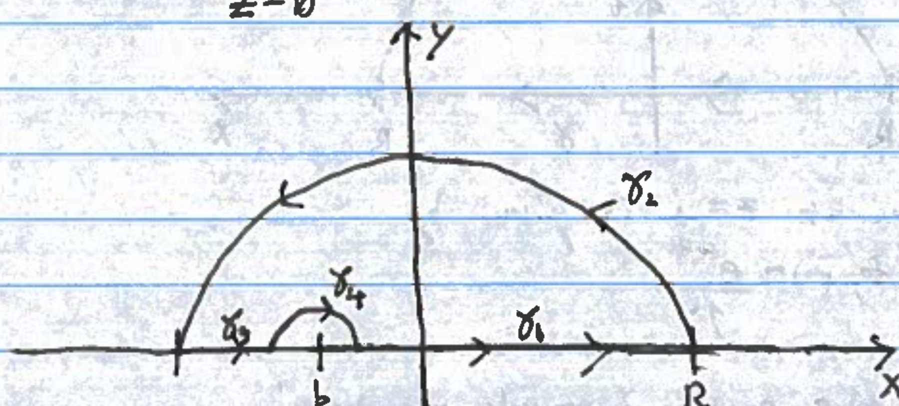
Compute each of the following

(b) $\text{P.V.}\int_{-\infty}^{\infty} \frac{\sin(ax)}{x-b}\,dx$

(c) $\text{P.V.}\int_{-\infty}^{\infty} \frac{\sin(x)}{x(1+x^2)}\,dx.$

Solution:

(a) Let $f(z) = \frac{e^{iaz}}{z-b}$ and consider the contour drawn below:



γ_2:

$$\left|\int_{\gamma_2} f(z)\,dz\right| = \left|\int_0^{\pi} \frac{e^{iaRe^{i\theta}}}{Re^{i\theta}-b} Rie^{i\theta}\,d\theta\right|$$

$$\leq \int_0^{\pi} \frac{e^{-aR\sin\theta}R}{R-b}\,d\theta$$

$$\leq \frac{C}{R-b}.$$

Therefore, by squeeze theorem

$$\lim_{R\to\infty}\int_{\gamma_2} f(z)\,dz = 0.$$

$\underline{\gamma_4:}$ $z = b+\delta e^{-i\theta}$, $\theta\in[-\pi,0]$.

$$\int_{\gamma_4} f(z)\,dz = \int_{-\pi}^{0} \frac{e^{ia(b+\delta e^{-i\theta})}}{\delta e^{-i\theta}}(-i\delta e^{-i\theta})\,d\theta.$$

$$= \int_{-\pi}^{0} -i e^{iab} e^{ia\delta e^{-i\theta}}\,d\theta.$$

Therefore,

$$\lim_{\delta\to 0}\int_{\gamma_4} f(z)\,dz = -i\pi e^{iab}.$$

Putting all the pieces together we have

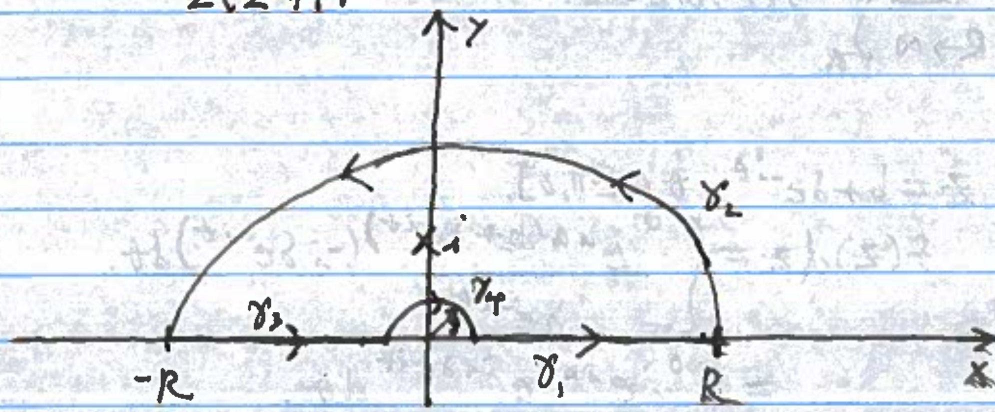
$$0 = \lim_{R\to\infty}\lim_{\delta\to 0}\int_{\gamma_4} f(z)\,dz = \text{P.V.}\int_{-\infty}^{\infty}\frac{e^{iaz}}{x-b}\,dz - i\pi e^{iab}.$$

Therefore,

$$i\pi(\cos(ab)+i\sin(ab)) = \text{P.V.}\int_{-\infty}^{\infty}\frac{\cos(ax)}{x-b}\,dx + i\,\text{P.V.}\int_{-\infty}^{\infty}\frac{\sin(ax)}{x-b}\,dx$$

$$\Rightarrow \text{P.V.}\int_{-\infty}^{\infty}\frac{\sin(ax)}{x-b}\,dx = \pi\cos(ab).$$

(c) Let $f(z) = \dfrac{e^{iz}}{z(z^2+1)}$ and consider the contour drawn below.



The residue at $z = i$ is given by

$$\operatorname{Res}(f; i) = \left.\frac{e^{iz}}{z(z+i)}\right|_{z=i} = \frac{e^{-1}}{-2} = -\frac{1}{2e}.$$

$\underline{\gamma_2:}$

$$\left|\int_{\gamma_2} f(z)\,dz\right| \le \int_0^{\pi} \frac{e^{-R\sin\theta}}{R^2-1}\,d\theta \le \frac{1}{R(R^2-1)}$$

and thus by squeeze theorem

$$\lim_{R\to\infty}\int_{\gamma_2} f(z)\,dz = 0.$$

$\underline{\gamma_4:}$

$$\int_{\gamma_4} f(z)\,dz = \int_{-\pi}^{0} \frac{e^{i\delta e^{-i\theta}}}{\delta e^{-i\theta}(\delta^2 e^{-2i\theta}+1)}\,(-\delta i e^{-i\theta})\,d\theta$$

$$= \int_{-\pi}^{0} -i\,\frac{e^{i\delta e^{-i\theta}}}{(\delta^2 e^{-2i\theta}+1)}\,d\theta$$

and thus

$$\lim_{\delta\to 0}\int_{\gamma_4} f(z)\,dz = -i\pi.$$

Putting everything together we have that

$$2\pi i\left(\frac{-1}{2e}\right)=\lim_{R\to\infty}\lim_{\delta\to 0}\int_{\Gamma} f(z)\,dz$$

$$=P.V.\int_{-\infty}^{\infty}\frac{\cos(x)}{x(x^2+1)}\,dx+i\,P.V.\int_{-\infty}^{\infty}\frac{\sin(x)}{x(x^2+1)}\,dx-i\pi.$$

Therefore,

$$P.V.\int_{-\infty}^{\infty}\frac{\sin(x)}{x(x^2+1)}\,dx=\pi\left(1-\frac{1}{e}\right).$$

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#4

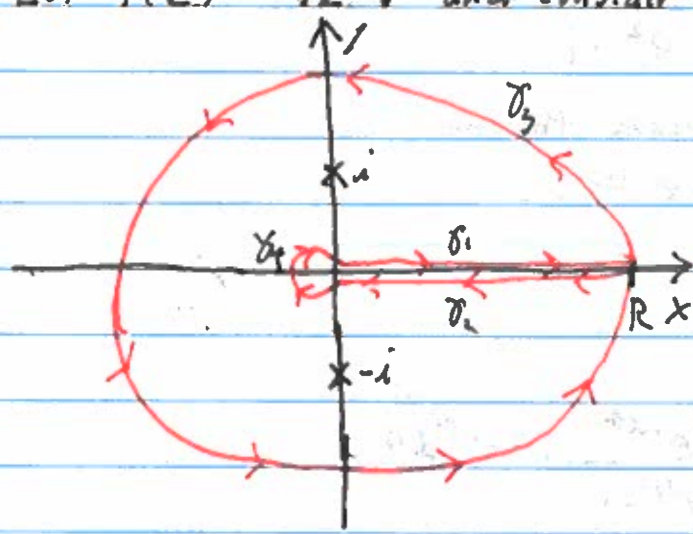
Compute each of the following

(a) $\displaystyle\int_0^{\infty}\frac{\sqrt{x}}{1+x}\,dx$

(b) $\displaystyle\int_0^{\infty}\frac{x^{\alpha}}{x^2-1}\,dx$

Solution:

(b) Let $f(z)=z^{\alpha}/z^2-1$ and consider the contour drawn below:



The residue at $z=i$

$$\operatorname{Res}(f;i)=\frac{i^\alpha}{2i}=\frac{e^{i\pi\alpha/2}}{2i},$$

and the residue at $z=-i$

$$\operatorname{Res}(f;-i)=\frac{(-i)^\alpha}{2i}=\frac{e^{3i\pi\alpha/2}}{2i}.$$

γ_1:

$$\int_{\gamma_1} f(z)\,dz=\int_0^R \frac{x^\alpha}{x^2-1}\,dx$$

γ_2:

$$\int_{\gamma_2} f(z)\,dz=\int_R^0 \frac{e^{2\pi i\alpha}x^\alpha}{x^2-1}\,dx=-\int_0^R \frac{e^{2\pi i\alpha}x^\alpha}{x^2-1}\,dx=-e^{2\pi i\alpha}\int_0^R \frac{x^\alpha}{x^2-1}\,dx.$$

γ_3:

$$\left|\int_{\gamma_3} f(z)\,dz\right|=\left|\int_0^{2\pi} \frac{R^\alpha e^{i\alpha\theta}}{R^2e^{-2i\theta}-1}\,iRe^{i\theta}\,d\theta\right|$$

$$\le\int_0^{2\pi}\frac{R^{1+\alpha}}{R^2-1}\,d\theta$$

Therefore, by the squeeze theorem

$$\lim_{R\to\infty}\int_{\gamma_3} f(z)\,dz=0.$$

γ_4:

$$\int_{\gamma_4} f(z)\,dz \int_{2\pi}^0 \frac{\delta^\alpha e^{i\alpha\theta}}{\delta^2 e^{-2i\theta}-1}\,i\delta e^{i\theta}\,d\theta$$

and this

$$\lim_{\delta\to 0}\int_{\gamma_4} f(z)\,dz=0.$$

Putting it all together we have that

$$\left(\frac{e^{i\pi\alpha/2}}{2i} + \frac{e^{3i\pi\alpha/2}}{2i}\right)2\pi i = \lim_{R\to\infty}\lim_{\delta\to 0}\int f(z)\,dz$$

$$= \left(1 - e^{2\pi i\alpha}\right)\int_0^\infty \frac{x^\alpha}{x^2-1}\,dx.$$

Therefore,

$$2\pi i\,\cancel{e^{i\pi\alpha}}\left(\frac{e^{i\pi\alpha/2} + e^{-i\pi\alpha/2}}{2i}\right) = \cancel{e^{i\pi\alpha}}\left(e^{\pi i\alpha} - e^{-i\pi\alpha}\right)\int_0^\infty \frac{x^\alpha}{x^2-1}\,dx$$

$$\Rightarrow 2\pi i\cos\left(\frac{\alpha}{2}\right) = 2i\sin(\pi\alpha)\int_0^\infty \frac{x^\alpha}{x^2-1}\,dx$$

$$\Rightarrow \int_0^\infty \frac{x^\alpha}{x^2-1}\,dx = \frac{\cos(\alpha/2)}{\sin(\pi\alpha)}.$$