MTH 317/617
Homework #1

Due Date: September 02, 2022

1 Problems for Everyone

1. Write the following complex expressions in the form $a + bi$ where a and b are real numbers.

   (a) $(-1+i)^2$,

   (b) $\dfrac{(8+2i)-(1-i)}{(2+i)^2}$,

   (c) $\dfrac{2+3i}{1+2i} - \dfrac{8+i}{6-i}$,

   (d) $\left(\dfrac{2+i}{6i-(1-2i)}\right)^2$.

2 2. Solve the following equations for z. Express your answer in the form $z = a + bi$ where a and b are real numbers.

   (a) $iz = 4 - zi$,

   (b) $\dfrac{z}{1-z} = 1 - 5i$,

   (c) $(2-i)z + 8z^2 = 0$,

   (d) $z^2 + 16 = 0$.

2 3. Let $z \in \mathbb{C}$ and assume $z \neq 0$. Prove the following:

   ½ (a) $|\mathrm{Re}(z)| \leq |z|$ and $|\mathrm{Im}(z)| \leq |z|$, (-¼) for writing

   ½ (b) $\mathrm{Re}(z) = \dfrac{z+\bar{z}}{2}$ and $\mathrm{Im}(z) = \dfrac{z-\bar{z}}{2i}$, (-¼)

   ½ (c) If k is an integer then $(\bar{z})^k = \overline{(z^k)}$, (-¼)

   ½ (d) If $|z| = 1$ and $z \neq 1$, then $\mathrm{Re}\left((1-z)^{-1}\right) = \dfrac{1}{2}$. (-¼)

4. Describe the set of point $z \in \mathbb{C}$ that satisfy each of the following.

   (a) $|z - 1 + 1| = 3$,

   (b) $|z - 1| = |z + 1|$,

   (c) $|z| = \mathrm{Re}(z) + 2$,

   (d) $2 < |z| < 6$.

5. Find the argument of the following complex numbers and write each in the polar form $z = r(\cos(\theta) + i\sin(\theta))$.

   (a) $-\frac{1}{2}$,

   (b) $-3+3i$,

   (c) $-\pi i$,

   (d) $-2\sqrt{3} - 2i$.

6. Write the given complex number in the form $a + bi$, where $a, b \in \mathbb{R}$.

   (a) $e^{-i\frac{\pi}{2}}$,

   (b) $\dfrac{e^{1+3\pi i}}{e^{-1+\frac{\pi i}{2}}}$,

   (c) $\dfrac{e^{3i} - e^{-3i}}{2i}$,

   (d) e^{e^i}.

2 Graduate Problems

1. Let B be an $m \times n$ matrix with complex valued entries. By $B^\dagger$ we denote the $n \times m$ matrix obtained by forming the transpose of B followed by taking the conjugate of each entry. Let A be an $n \times n$ matrix with complex entries. Prove that if $\mathbf{u}^\dagger A \mathbf{u} = 0$ for all $n \times 1$ column vectors $\mathbf{u}$ with complex entries, then A is the 0 matrix.

Homework #1

#2.

Solve the following equations for z. Express your answer in standard form.

(a) $iz = 4 - zi$

(b) $z/1-z = 1-5i$

(c) $(2-i)z + 8z^2 = 0$

(d) $z^2 + 16 = 0$.

Solution:

(a) $iz = 4 - zi$

$\Rightarrow 2iz = 4$

$\Rightarrow z = 2/i = -2i$.

(b) $\dfrac{z}{1-z} = 1-5i$

$\Rightarrow z = (1-5i)(1-z)$

$\Rightarrow \cancel{z = (1-5i)z + 1 - 5i}$ $\Rightarrow z = (1-5i) - (1-5i)z$

$\Rightarrow \cancel{5iz = 1-5i}$ $\Rightarrow (1-5i+1)z = 1-5i$

$\Rightarrow \cancel{z = \frac{1}{5i} - 1}$ $\Rightarrow z = \dfrac{1-5i}{2-5i}$

$\Rightarrow \cancel{z = -1 - \frac{1}{5}i}$. $\Rightarrow z = \dfrac{(1-5i)(2+5i)}{(2-5i)(2+5i)}$ ... $\Rightarrow z = \dfrac{27}{29} - \dfrac{5}{29}i$

(c) $(2-i)z + 8z^2 = 0$

$\Rightarrow z(2-i+8z) = 0$

$\Rightarrow z = 0, -1/4 + i/8$.

(d) $z^2 + 16 = 0$

$\Rightarrow z = \pm 4i$

■

#3

Let $z \in \mathbb{C}$ and assume $z \neq 0$. Prove the following

(a) $|Re(z)| \leq |z|$ and $|Im(z)| \leq |z|$,

(b) $Re(z) = \frac{z+\bar{z}}{2}$, $Im(z) = \frac{z-\bar{z}}{2i}$,

(c) If k is an integer then $(\bar{z})^k = \overline{(z^k)}$,

(d) If $|z|=1$ and $z \neq 1$ then $Re((1-z)^{-1}) = \frac{1}{2}$.

~~proof~~:

(a) For all $z \in \mathbb{C}$,
$$z = Re(z) + i\,Im(z)$$
and thus
$$|z| = \sqrt{Re(z)^2 + Im(z)^2}$$
$$\Rightarrow |z| \geq |Re(z)| \text{ and } |z| \geq |Im(z)|.$$

(b) For all $z \in \mathbb{C}$,
$$z = Re(z) + i\,Im(z).$$
Therefore,
$$\frac{z+\bar{z}}{2} = \frac{Re(z)+i\,Im(z)+Re(z)-i\,Im(z)}{2} = Re(z).$$
$$\frac{z-\bar{z}}{2i} = \frac{Re(z)+i\,Im(z)-Re(z)+i\,Im(z)}{2i} = Im(z).$$

(c) Let $z \in \mathbb{C}$ and $k \in \mathbb{Z}$. It follows from De Moivre's formula that there exists $\theta \in (-\pi, \pi]$ such that
$$\bar{z}^k = \overline{|z|(\cos\theta + i\sin\theta)}^k$$
$$= |z|(\cos\theta - i\sin\theta)^k$$
$$= |z|(\cos(-\theta) + i\sin(-\theta))^k$$
$$= |z|(\cos(-k\theta) + i\sin(-k\theta))$$
$$= |z|(\cos(k\theta) - i\sin(k\theta))$$
$$= \overline{z^k}.$$

(d) Let $z \in \mathbb{C}$ and satisfy $z \neq 1$, $|z|=1$. Therefore, there exists $\theta \in (-\pi, \pi]$ such that $z = e^{i\theta}$. Consequently,
$$\frac{1}{1-z} = \frac{1}{1-e^{i\theta}} = \frac{1}{1-\cos\theta - i\sin\theta} = \frac{1-\cos\theta + i\sin\theta}{(1-\cos\theta)^2 + \sin^2\theta} = \frac{1-\cos\theta + i\sin\theta}{2-2\cos\theta}$$
$$\Rightarrow Re\left(\frac{1}{1-z}\right) = \frac{1}{2}$$
$\blacksquare$

#4.

Describe the set of points $z \in \mathbb{C}$ that satisfy each of the following.

(a) $|z-1+1|=3$

(b) $|z-1|=|z+1|$

(c) $|z|=\mathrm{Re}(z)+2$

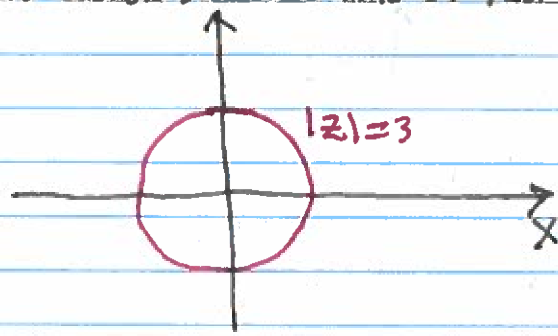
(d) $2<|z|<6$

Solution:

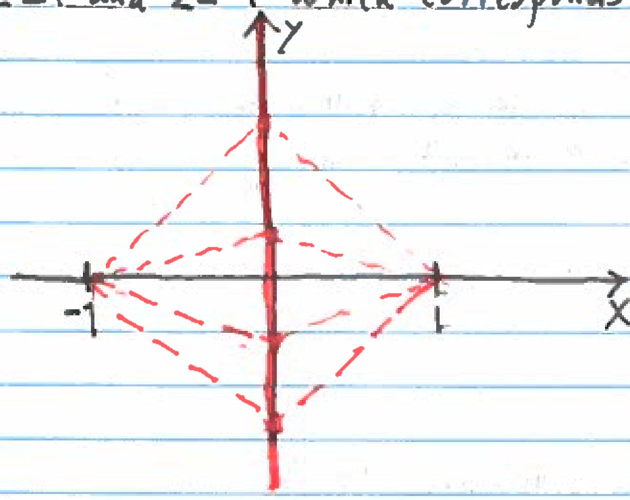
(a) $|z-1+1|=3$

$\Rightarrow |z|=3$

This describes a circle of radius $\sqrt{3}$ centered at the origin.



(b) $|z-1|=|z+1|$ describes the set of points equidistant from $z=1$ and $z=-1$ which corresponds to the imaginary axis.



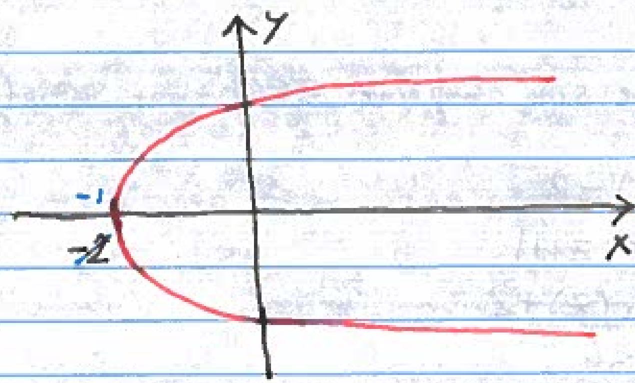
(c) Algebraically, $|z|=\mathrm{Re}(z)+2$ is equivalent to

$$\sqrt{x^2+y^2}=x+2$$

$$\Rightarrow x^2+y^2=x^2+\overset{4}{2}x+4$$

$$\Rightarrow y^2=\overset{4}{2}x+4$$

which corresponds to a rightward opening parabola:



(d) The set $2<|z|<6$ describes an annulur region with inner radius $\sqrt{2}$ and outer radius $\sqrt{6}$ and centered at the origin.

#5

Find the argument of the following complex numbers and write each in the polar form $z=r(\cos\theta+i\sin\theta)$.

(a) $-\frac{1}{2}$

(b) $-3+3i$

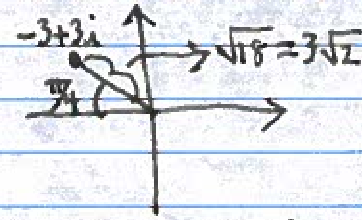
(c) πi

(d) $-2\sqrt{3}-2i$

Solution:

(a) $-\frac{1}{2}=\frac{1}{2}(\cos(\pi)+i\sin(\pi))$

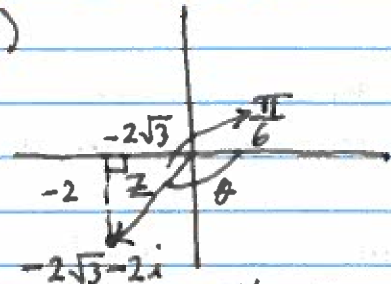
(b)



$-3+3i=3\sqrt{2}(\cos(3\pi/4)+i\sin(3\pi/4))$

(c) $-\pi i = \pi(-i) = \pi(\cos(-\pi/2) + i\sin(-\pi/2))$.

(d)



$|z| = (4\cdot 3 + 4)^{1/2} = 4$

$\theta = -5\pi/6$

Therefore,

$z = 4(\cos(-5\pi/6) + i\sin(-5\pi/6))$.

#6.

Write the given complex number in the form $a+bi$, where $a, b \in \mathbb{R}$.

(a) $e^{-i\pi/2}$

(b) $\dfrac{e^{1+3\pi i}}{e^{-1+\pi i/2}}$

(c) $\dfrac{e^{3i}-e^{-3i}}{2i}$

(d) $e^{e^{i}}$

Solution:

(a) $e^{-i\pi/2} = \cos(-\pi/2) + i\sin(-\pi/2) = -i$.

(b) $\dfrac{e^{1+3\pi i}}{e^{-1+\pi i/2}} \cancel{= e^{3\pi i - \pi i/2} = e^{5\pi i/2} = \cos(5\pi/2) + i\sin(5\pi/2) = i}$.

$= e^{\wedge}(1+3\pi i - (-1+\pi i/2)) = e^{\wedge}(2 + \frac{5}{2}\pi i) = e^2 e^{\frac{5}{2}\pi i}$

$= e^2(\cos(\frac{5}{2}\pi) + i\sin(\frac{5}{2}\pi))$

$= ie^2$

(c) $\dfrac{e^{3i}-e^{-3i}}{2i} = \sin(3)$

(d) $e^{e^{i}} = e^{\cos(1)+i\sin(1)} = e^{\cos(1)}e^{i\sin(1)} = e^{\cos(1)}(\cos(\sin(1)) + i\sin(\sin(1)))$

Graduate Problems

#1

Let A be an $n \times n$ matrix with complex entries. Prove that if $u^* A u = 0$ for all $n \times 1$ complex vectors, then A is the 0 matrix.

proof:

In component form, $u^* A u = 0$ is equivalent to the equation

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \bar{u}_i A_{ij} u_j = 0.$$

Therefore, if we let $u_i = 0$ unless $i = k, \ell$ we obtain the equation

$$\bar{u}_k A_{kk} u_k + \bar{u}_k A_{k\ell} u_\ell + \bar{u}_\ell A_{\ell k} u_k + \bar{u}_\ell A_{\ell\ell} u_\ell = 0.$$

If we choose $u_k = 0$ and $u_\ell = 1$ it follows that

$$A_{\ell\ell} = 0,$$

i.e. all diagonal entries are 0. If we let $u_k = 1$ and $u_\ell = 1$ it follows that

$$A_{\ell k} + A_{k\ell} = 0$$

$$\Rightarrow A_{\ell k} = -A_{k\ell}$$

Therefore,

$$\bar{u}_k A_{k\ell} u_\ell - \bar{u}_\ell A_{k\ell} u_k = 0$$

Now, let $u_\ell = 1 + i$ and $u_k = 1$. It follows that

$$(1+i) A_{k\ell} - (1-i) A_{k\ell} = 0$$

$$\Rightarrow 2i A_{k\ell} = 0$$

$$\Rightarrow A_{k\ell} = 0. \qquad \blacksquare$$

MTH 351/651
Homework #1

Due Date: September 02, 2022

1 Problems for Everyone

2 1. Consider the system $\dot{x} = \sin(x)$.

(a) Find all fixed points of the flow.

(b) At which points x does the flow have the greatest velocity to the right?

(c) Find the flows acceleration $\ddot{x}$ as a function of x.

(d) Find the points where the flow has maximum positive acceleration.

1 2. For the following equations sketch the vector fields on the real line, if possible find all fixed points, classify their stability and sketch the graph of $x(t)$ for different initial conditions. You must include enough sketches of $x(t)$ to illustrate all qualitatively different solution solution curves.

(a) $\dot{x} = 1 - x^{14}$.

(b) $\dot{x} = e^{-x}\sin(x)$.

(c) $\dot{x} = 1 - 2\cos(x)$.

(d) $\dot{x} = e^{x} - \cos(x)$ (You won't be able to find the fixed points explicitly, but you can still determine the qualitative behavior).

3 3. pg. 37: #2.2.8, 2.2.9, 2.2.10.

2 4. The velocity $v(t)$ of a skydiver falling to the ground is governed by the equation $m\dot{v} = mg - kv^2$, where m is the mass of the skydiver, g is the acceleration due to gravity, and $k > 0$ is a constant related to air resistance.

(a) Obtain the analytic solution for $v(t)$, assuming that $v(0) = 0$. 1/2

(b) Find the limit of $v(t)$ as $t \to \infty$. This limiting velocity is called the terminal velocity. 1/2

(c) Give a graphical analysis of this problem, and thereby re-derive a formula for the terminal velocity. 1

2 5. Suppose X and Y are two species that reproduce exponentially fast: $\dot{X} = aX$ and $\dot{Y} = bY$, respectively, with initial conditions $X_0, Y_0 > 0$ and growth rates $a, b > 0$. Let $x(t) = X(t)/(X(t) + Y(t))$ denotes X's share of the total population.

(a) Show that $\dot{x} = (a - b)x(1 - x)$.

(b) Show that if $a > b$ then x is monotonically increasing and approaches 1 as $t \to \infty$. What does this result imply about the population?

(c) Show that if $a < b$ then x is monotonically decreasing and approaches 0 as $t \to \infty$. What does this result imply about the population?

(d) What happens if $a = b$?

Homework #1

#1

Consider the system $\dot{x} = \sin(x)$.

(a) Find all fixed points of the flow.

(b) At which points x does the flow have the greatest velocity to the right?

(c) Find the flows acceleration $\ddot{x}$ as a function of x.

(d) Find the points where the flow has maximum positive acceleration.

Solution:

(a) The fixed points satisfy $\sin(x) = 0$, i.e., $x = n\pi$ where $n \in \mathbb{Z}$.

(b) The velocity $\dot{x}$ is greatest when $\sin(x)$ is maximized, i.e., $x = \pi/2 + 2n\pi$, where $n \in \mathbb{Z}$.

(c) Differentiating, it follows that

$$\ddot{x} = \frac{d}{dt}\dot{x} = \frac{d}{dt}\sin(x) = \cos(x)\cdot\dot{x} = \cos(x)\sin(x) = \tfrac{1}{2}\sin(2x).$$

(d) The flow has maximum acceleration when $\frac{1}{2}\sin(2x)$ is maximized, i.e., $2x = \pi/2 + 2n\pi$ which implies

$$x = \pi/4 + n\pi,$$

where $n \in \mathbb{Z}$.

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#2

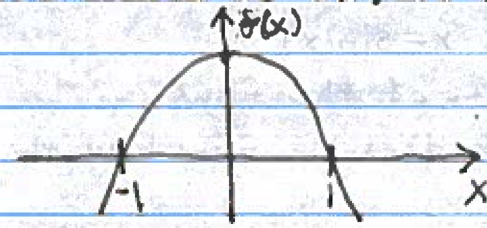
For the following equations sketch the vector fields on the real line, if possible find all fixed points, classify their stability and sketch the graph of $x(t)$ for different initial conditions.

(a) $\dot{x} = 1 - x^{14}$

(b) $\dot{x} = e^{-x}\sin(x)$

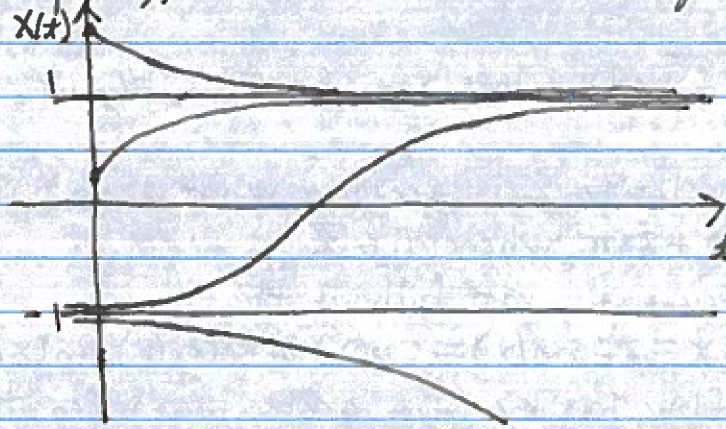
Solution:

(a) The sketch of $f(x)=1-x^{14}$ is given below

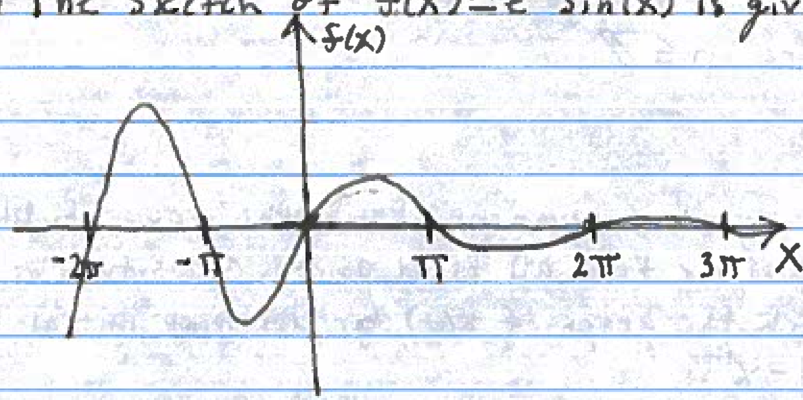


and thus the phase portrait is given by:

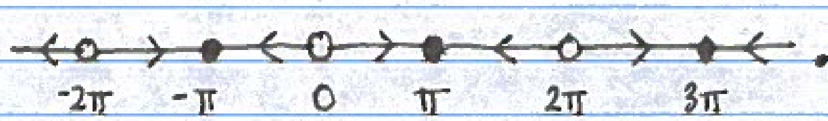
Consequently, the solution curves are given by:



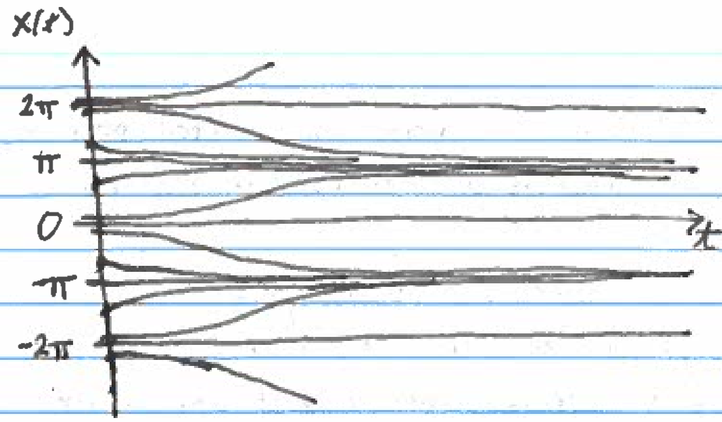
(b) The sketch of $f(x)=e^{-x}\sin(x)$ is given below



and thus the phase portrait is given by:

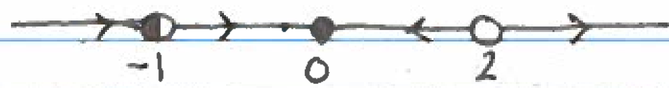


Consequently, the solution curves are given by



pg. 37, #2.2.8

For the phase portrait shown below, find an equation $\dot{x}=f(x)$ that is consistent with it.



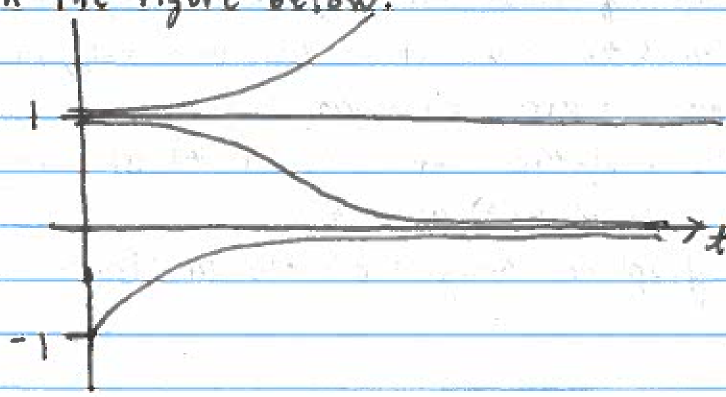
Solution:

One such equation is given by:

$$f(x) = (x+1)^2 x(x-2)$$

pg. 37, #2.2.9

Find an equation $\dot{x}=f(x)$ whose solutions $x(t)$ are consistent with the figure below.



Solution:

One such equation is given by:

$$f(x) = -x(1-x) = x(x-1).$$

~~pg.~~ 37, #2.2.10

Find an equation $\dot{x}=f(x)$ with the stated properties, or if there are no examples, explain why not. In all cases, assume that $f(x)$ is smooth.

(a) Every real number is a fixed point.

(b) Every integer is a fixed point, and there are no others.

(c) There are precisely three fixed points, and all of them are stable.

(d) There are no fixed points.

(e) There are precisely 100 fixed points.

Solution:

(a) The function $f(x)=0$ works.

(b) The function $f(x)=\sin(\pi x)$ works.

(c) There is no smooth function that works.

(d) The function $f(x)=1$ works.

(e) The function $f(x)=(x-1)(x-2)\cdots(x-100)$ works.

#4

The velocity $v(t)$ of a skydiver falling to the ground is governed by the equation $m\dot{v}=mg-Kv^2$, where m is the mass of the skydiver, g is the acceleration due to gravity, and $K>0$ is a constant related to air resistance.

(a) Obtain the analytic solution for $v(t)$, assuming $v(0)=0$.

(b) Find the limit of $v(t)$ as $t\to\infty$.

(c) Give a graphical analysis of the problem.

Solution:

(a) Integrating, it follows that

$$\int_0^{v(t)} \frac{m}{mg-Kv^2}\,dv=t$$

$$\Rightarrow \frac{1}{g}\int_0^{v(t)} \frac{1}{1-\frac{K}{gm}v^2}\,dv=\frac{1}{g}\sqrt{\frac{gm}{K}}\tanh^{-1}\left(\sqrt{\frac{K}{gm}}\,v\right)=t$$

$$\Rightarrow \sqrt{\frac{K}{gm}}\, v = \tanh(\sqrt{gK}\, t)$$

$$\Rightarrow v = \sqrt{\frac{gm}{K}} \tanh\left(\frac{\sqrt{gK}\, t}{\sqrt{m}}\right)$$

(b) Computing the limit, we have that

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \sqrt{\frac{gm}{K}} \tanh\left(\frac{\sqrt{gK}\, t}{\sqrt{m}}\right) = \sqrt{\frac{gm}{K}}.$$

(c) Sketching the phase portrait we have that the only fixed point is $v^* = \sqrt{gm/K}$ and it is stable.

and thus

$$\lim_{t\to\infty} v(t) = \sqrt{\frac{gm}{K}}.$$

#5.

Suppose X and Y are two species that reproduce exponentially fast: $\dot{X} = aX$, $\dot{Y} = bY$ with initial conditions $X_0, Y_0 > 0$ and growth rates $a, b > 0$. Let $x = X/(X+Y)$ denote X's share of the population.

(a) Show that $\dot{x} = (a-b)x(1-x)$.

(b) Show that if $a > b$ then x is monotonically increasing and approaches 1 as $t \to \infty$. What does this result imply about the population?

(c) Show that if $a < b$ then x is monotonically decreasing and approaches 0 as $t \to \infty$. What does this result imply about the population?

(d) What happens if $a = b$?

Solution:

(a) Differentiating, we have that

$$\dot{x} = \frac{(X+Y)\dot{X} - X(\dot{X}+\dot{Y})}{(X+Y)^2}$$

$$= \frac{\dot{X}Y - X\dot{Y}}{(X+Y)^2}$$

$$= \frac{aXY - bXY}{(X+Y)^2}.$$

$$\Rightarrow \dot{x} = \frac{X(a-b)Y}{(X+Y)^2}$$

$$= x(a-b)\frac{Y}{X+Y}.$$

Now, since $X+Y = X/x$ and $Y = X(1-x)/x$ it follows that

$$\dot{x} = \frac{x(a-b)}{X/x} \cdot \frac{X(1-x)}{x} = (a-b)\,x(1-x).$$

(b-d) There are three possibilities for the phase portraits.

$(a>b)$


$(a<b)$


$(a=b)$

If $a>b$ the x population dominates and becomes the most represented population. If $a<b$ the y population becomes over represented. If $a=b$ the ratio remains constant for all time.

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