MTH 317/617
Homework #2

Due Date: September 09, 2022

1 Problems for Everyone

1. Show that the function $z(t) = e^{it}$, $0 \leq t \leq 2\pi$, is the parametric representation of a the unit circle in the complex plane traversed in the counterclockwise direction.

2. Sketch the following curves in the complex plane:

   (a) $z(t) = 3e^{it}, 0 \leq t \leq 2\pi$.

   (b) $z(t) = 2e^{-it}, 0 \leq t \leq \pi$.

   (c) $z(t) = e^{-(1+i)t}, 0 \leq t \leq 2\pi$.

   (d) $z(t) = e^{(1+i)t}, 0 \leq t \leq 2\pi$.

3. Find all the values of the following. Express your answers in the form $z = a + ib$ where $a, b \in \mathbb{R}$.

   (a) $(-16)^{1/4}$.

   (b) $i^{1/4}$.

   (c) $(1 - \sqrt{3}i)^{1/3}$.

   (d) $\left(\dfrac{2i}{1+i}\right)^{1/6}$.

   (e) i^i.

4. Let $a_0, \ldots, a_n \in \mathbb{R}$ and assume $z^* \in \mathbb{C}$ is a root of the polynomial

$$p(z) = a_0 + a_1 z + \ldots a_n z^n.$$

   (a) Prove that for all $z \in \mathbb{C}$, $p(\overline{z}) = \overline{p(z)}$.

   (b) Prove that $\overline{z^*}$ is also a root of the polynomial.

5. Sketch each of the following sets and determine which of the sets are *domains:*

   (a) $\{z \in \mathbb{C} : |z - 1 + i| \leq 3\}$

   (b) $\{z \in \mathbb{C} : 0 < |z - 2| < 3\}$

   (c) $\{z \in \mathbb{C} : |z| \geq 2\}$

   (d) $\{z \in \mathbb{C} : -1 < \mathrm{Im}(z) \leq 1\}$

   (e) $\{z \in \mathbb{C} : (\mathrm{Re}(z))^2 > 1\}$

6. For each of the following points in $\mathbb{C}$, determine its stereographic projection on the Riemann sphere

   (a) $z = i$

   (b) $z = 6 - 8i$

   (c) $z = -\frac{3}{10} + \frac{2}{5}i$.

7. Describe the projections on the Riemann sphere of the following sets in the complex plane $\mathbb{C}$:

   (a) The right half plane $\{z : \text{Re}(z) > 0\}$

   (b) The disk $\{z : |z| < 1/2\}$

   (c) The annulus $\{z : 1 < |z| < 2\}$

   (d) The set $\{z : |z| > 3\}$

   (e) The line $y = x$ (including the point at ∞).

2 Graduate Problems

1. The following problems are related to the topology of $\mathbb{C}$.

   (a) If S_i is a countably infinite collection of open subsets of $\mathbb{C}$ indexed by $i \in \mathbb{N}$, prove that $\bigcup_{i=1}^{\infty} S_i$ is open.

   (b) If S_i is a finite collection of open subsets of $\mathbb{C}$ indexed by $i \in \{1, \ldots, n\}$, prove that $\bigcap_{i=1}^{n} S_i$ is open.

   (c) If S_i is a countably infinite collection of open subsets of $\mathbb{C}$ indexed by $i \in \mathbb{N}$, then $\bigcap_{i=1}^{\infty} S_i$ is not necessarily open. Show this by constructing an explicit example.

   (d) Prove that if S and T are domains in $\mathbb{C}$ with at least one point in common, then $S \cup T$ is a domain.

   (e) If S and T are domains, is $S \cap T$ a domain? If so, prove it. It not, draw a counterexample.

#2.

Sketch the following curves in $\mathbb{C}$.

(a) $z(t)=3e^{it}$, $0\leq t\leq 2\pi$

(b) $z(t)=2e^{-it}$, $0\leq t\leq \pi$

(c) $z(t)=e^{-(1+i)t}$, $0\leq t\leq 2\pi$

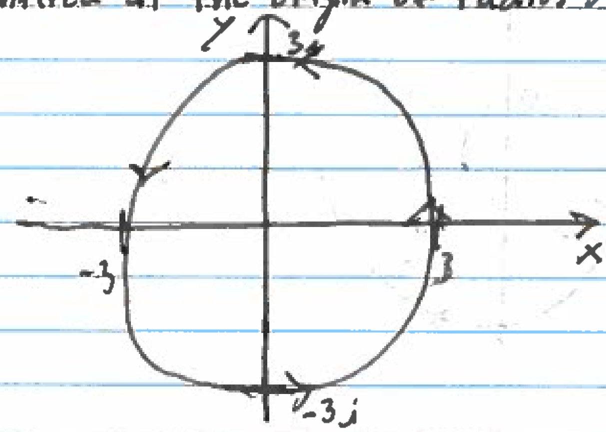
(d) $z(t)=e^{(1+i)t}$, $0\leq t\leq 2\pi$

Solution:

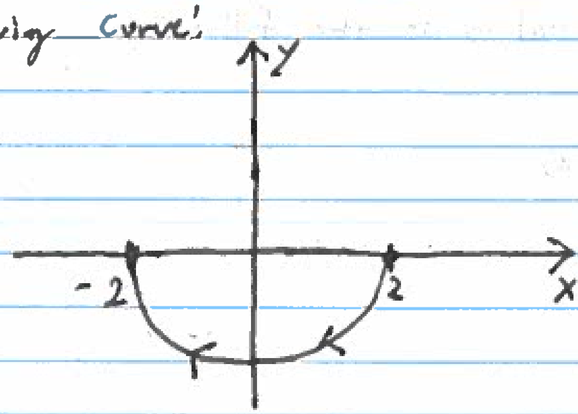
(a) $z(t)=3e^{it}=3(\cos(t)+i\sin(t))$ and thus

$$x(t)=3\cos(t),$$
$$y(t)=3\sin(t),$$

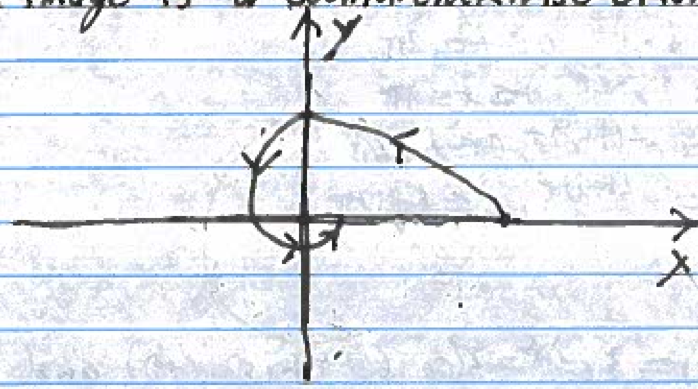
which is the parametrization of counterclockwise circle centered at the origin of radius 3.



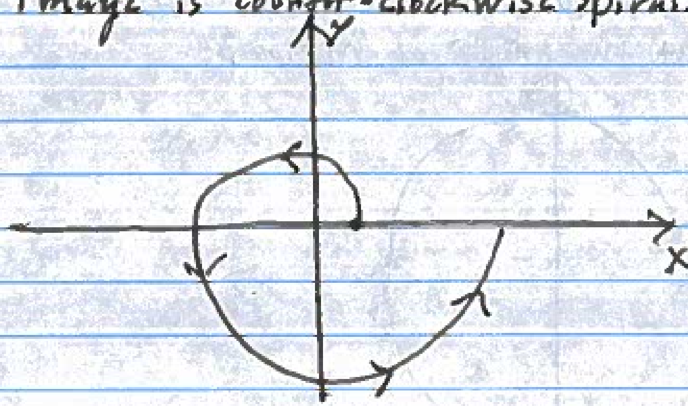
(b) $z(t)=2e^{-it}=2(\cos(t)-i\sin(t))$ which yields the following curve:



(c) $z(t) = e^{-(1+i)t} = e^{-t}e^{-it} = e^{-t}(\cos(t) + i\sin(t))$ and thus the image is a counterclockwise oriented spiral.



(d) $z(t) = e^{(1+i)t} = e^{t}e^{it} = e^{t}(\cos(t) + i\sin(t))$ and thus the image is counter-clockwise spiral.



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#3

Find all the values of the following

(a) $(-16)^{1/4}$

(b) $i^{1/4}$

(c) $(1-\sqrt{3}\,i)^{1/3}$

(d) $\left(\frac{2i}{1+i}\right)^{1/6}$

(e) i^{i}.

Solution:

(a) $(-16)^{1/4} = (16e^{i\pi+2n\pi i})^{1/4} = 2e^{i\pi/4+n\pi i/2} = 2e^{i\pi/4(1+2n)}$ and thus

$$(-16)^{1/4} = 2e^{i\pi/4}, 2e^{3i\pi/4}, 2e^{5i\pi/4}, 2e^{7i\pi/4}.$$

(b) $i^{1/4} = (e^{i\pi/2+2n\pi})^{1/4} = (e^{\frac{i\pi(1+4n\pi)}{2}})^{1/4} = e^{i\pi(1+4n\pi)/8}$ and thus

$$i^{1/4} = e^{i\pi/8}, e^{i5\pi/8}, e^{i9\pi/8}, e^{i13\pi/8}$$

(c) $(1-\sqrt{3})^{1/3} = (2e^{-i\pi/3+2n\pi i})^{1/3} = (2e^{-i\pi(1-6n\pi i)/3})^{1/3} = 2^{1/3}e^{-i\pi/6(1-6n\pi i)}$

and thus

$$(1-\sqrt{3})^{1/3} = 2^{1/3}e^{-i\pi/6}, 2^{1/3}e^{5n\pi i/6}, 2^{1/3}e^{11n\pi i/6}, 2^{1/3}e^{17n\pi i/6}$$

(d) $\left(\frac{2i}{1+i}\right)^{1/6} = \left(\frac{2i(1-i)}{2}\right)^{1/6} = (1+i)^{1/6} = (\sqrt{2}e^{i\pi/4+2n\pi i})^{1/6} = 2^{1/12}e^{i\pi(1+8n)/24}$

and thus

$$\left(\frac{2i}{1+i}\right)^{1/6} = 2^{1/12}e^{i\pi/24}, 2^{1/12}e^{9\pi i/24}, 2^{1/12}e^{17\pi i/24}, 2^{1/12}e^{25\pi i/24}, 2^{1/12}e^{33\pi i/24}, 2^{1/12}e^{41\pi i/24}.$$

(e) $i^i = (e^{i\pi/2+2n\pi i})^i = e^{-\pi/2-2n\pi}$ and thus

$$i^i = \ldots e^{-\pi/2-2\pi}, e^{-\pi/2}, e^{-\pi/2+2\pi}, e^{-\pi/2+4\pi}, \ldots$$

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#4.

Let $a_0, \ldots, a_n \in \mathbb{R}$ and assume $z^* \in \mathbb{C}$ is a root of the polynomial

$$p(z) = a_0 + a_1 z + \ldots + a_n z^n.$$

(a) Prove that for all $z \in \mathbb{C}$, $p(\bar{z}) = \overline{p(z)}$.

(b) Prove that $\overline{z^*}$ is also a root of the polynomial.

Solution:

(a) Computing it follows that

$$\begin{aligned} p(\bar{z}) &= a_0 + a_1\bar{z} + \ldots + a_n\bar{z}^n \\ &= \overline{a_0} + \overline{a_1 z} + \ldots + \overline{a_n z^n} \\ &= \overline{a_0 + a_1 z + \ldots + a_n z^n} \\ &= \overline{p(z)}. \end{aligned}$$

(b) If z^* is a root then

$$p(\overline{z^*}) = \overline{p(z^*)} = \bar{0} = 0.$$

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#5

Sketch each of the following sets and determine which are domains:

(a) $\{z \in \mathbb{C} : |z-1+i| \leq 3\}$

(b) $\{z \in \mathbb{C} : 0 < |z-2| < 3\}$

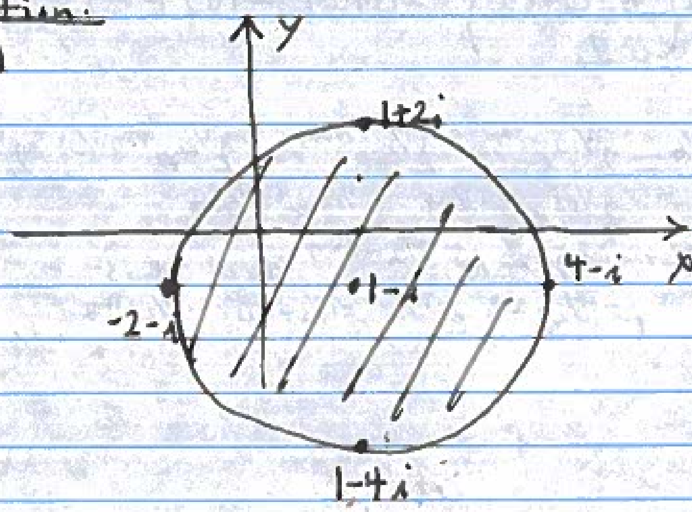
(c) $\{z \in \mathbb{C} : |z| \geq 2\}$

(d) $\{z \in \mathbb{C} : -1 < \mathrm{Im}(z) \leq 1\}$

(e) $\{z \in \mathbb{C} : (\mathrm{Re}(z))^2 > 1\}$.

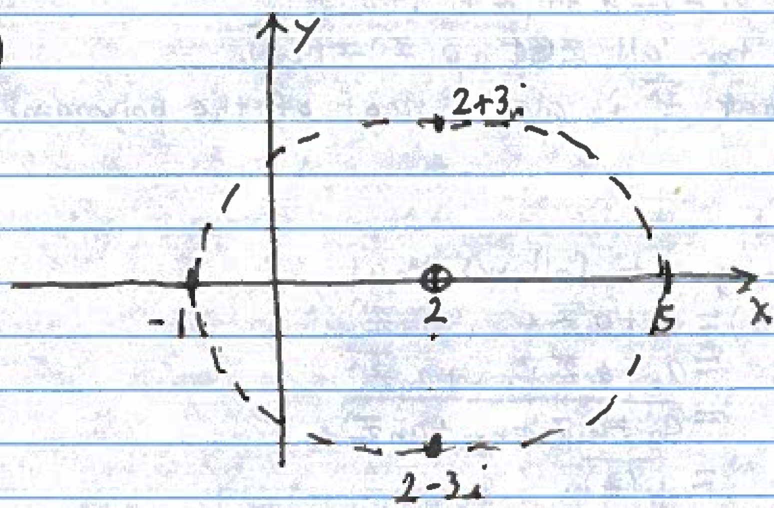
Solution:

(a)



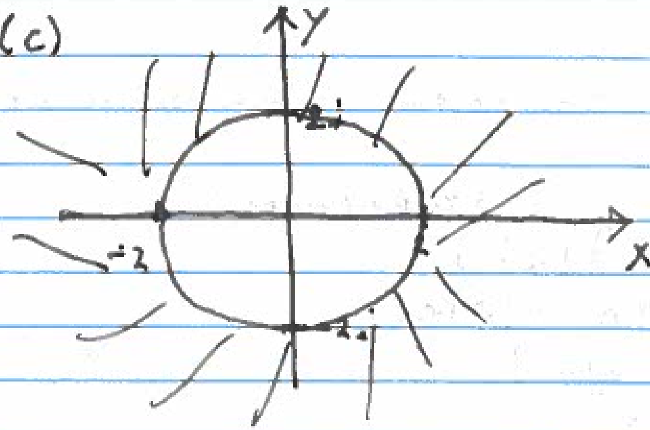
This set is not a domain.

(b)



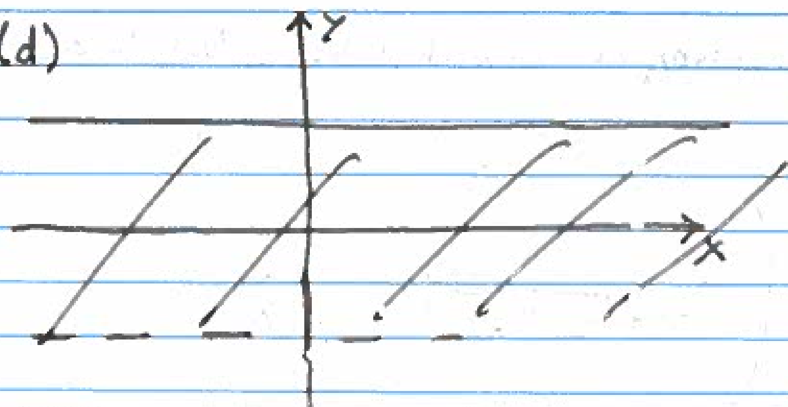
This set is a domain.

(c)



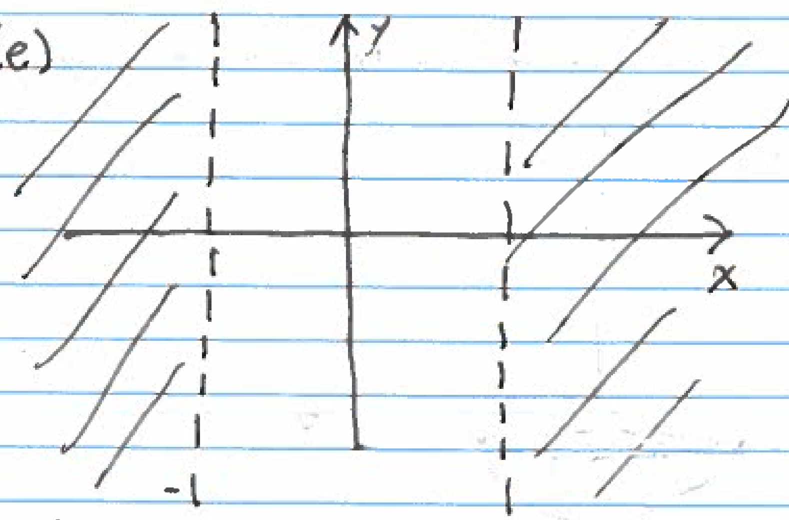
This set is a domain.

(d)



This set is not a domain.

(e)



This set is a domain.

#7.

Describe the projections on the Riemann sphere of the following sets in $\mathbb{C}$.

(a) The right half plane $\{z : \mathrm{Re}(z) > 0\}$.

(b) The disk $\{z : |z| < 1/2\}$.

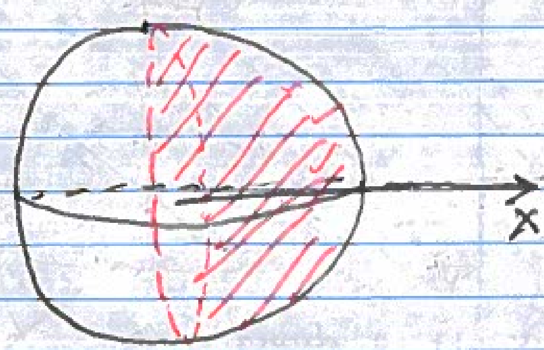
(c) The annulus $\{z : 1 < |z| < 2\}$.

(d) The set $\{z : |z| > 3\}$.

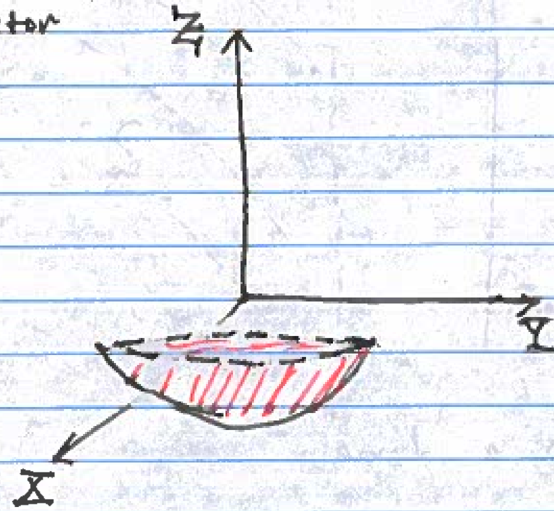
(e) The line $y = x$

Solution:

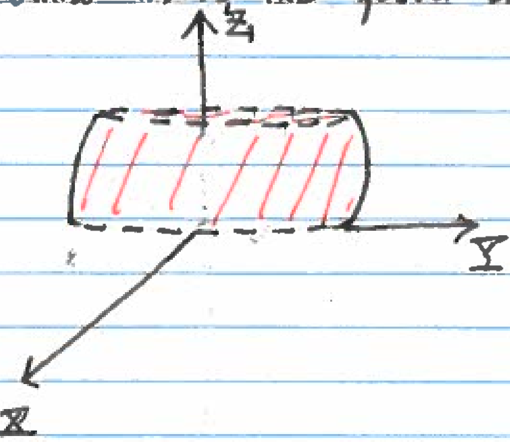
(a) The hemisphere described by $X^2 + Y^2 + Z^2 = 1$ and $X > 0$:



(b) A hemisphere centered at the south pole lying below the equator



(c) A band above the equator and below the north pole



(d) A hemisphere covering the north pole but above the equator.