MTH 317/617
Homework #3

Due Date: September 16, 2022

1 Problems for Everyone

1. Write each of the following functions in the form $w = u(x, y) + iv(x, y)$ and for each function find the domain of definition.

   (a) $f(z) = 3z^2 + 5z + i + 1$

   (b) $f(z) = 1/z$

   (c) $f(z) = \dfrac{z + i}{z^2 + 1}$

   (d) $f(z) = e^{3z}$

   (e) $f(z) = \dfrac{2z^2 + 3}{|z - 3|}$

   (f) $f(z) = e^z + e^{-z}$

2. For the complex function $f(z) = e^z$:

   (a) Describe the domain of definition and the range.

   (b) Show that $f(-z) = -1/f(z)$.

   (c) Describe the image of the vertical line $\text{Re}(z) = 1$.

   (d) Describe the image of the horizontal line $\text{Im}(z) = \pi/4$.

   (e) Describe the image of the infinite strip $0 \leq \text{Im}(z) \leq \pi/4$.

3. Let $F(z) = z + i$, $G(z) = iz$, and $H(z) = 2z$. Sketch the image of the semi-circle:

$$S = \{z \in \mathbb{C} : |z| = 1, \text{Im}(z) > 0 \text{ and } \text{Re}(z) > 0\}$$

   under the following mappings:

   (a) $F(z)$

   (b) $G(z)$

   (c) $H(z)$

   (d) $G(F(z))$

   (e) $G(H(z))$

   (f) $H(F(z))$

   (g) $F(G(H(z)))$

4. Prove the sequence of complex numbers $z_n = x_n + iy_n$ converges to $z_0 = x_0 + iy_0$ if and only if x_n converges to x_0 and y_n converges to y_0.

5. Prove that the sequence of complex numbers $z_n \to z_0$ if and only if $\overline{z_n} \to \overline{z_0}$.

6. Prove that $z_n \to 0$ if and only if $|z_n| \to 0$.

7. Compute the following limits justifying all steps or prove that the limit does not exist.

   (a) $\lim_{z \to 0} \frac{\mathrm{Im}(z)}{z}$.

   (b) $\lim_{z \to 0} z e^{i\mathrm{Re}(z)}$.

   (c) $\lim_{z \to 0} e^{\frac{1}{z}}$.

   (d) $\lim_{z \to i} \frac{1}{z-i} - \frac{1}{z^2+1}$.

Homework #3

#1.

Write each of the following functions in the form

$$w = u(x,y) + i v(x,y)$$

and for each function find the domain of definition.

(a) $f(z) = 3z^2 + 5z + i + 1$

(b) $f(z) = 1/z$

(c) $f(z) = \dfrac{z+i}{z^2+1}$

(d) $f(z) = e^{3z}$

(e) $f(z) = \dfrac{2z^2+3}{|z-3|}$

(f) $f(z) = e^z + e^{-z}$

Solution:

(a) The domain of definition is $\mathbb{C}$. Computing, we have that

$$\begin{aligned} f(z) &= 3(x+iy)^2 + 5(x+iy) + i + 1 \\ &= 3x^2 - 3y^2 + 6xiy + 5x + 5iy + i + 1 \\ &= 3x^2 - 3y^2 + 5x + 1 + i(6xy + 5y + 1) \\ &= u(x,y) + i v(x,y). \end{aligned}$$

(b) The domain of definition is $\mathbb{C} \setminus \{0\}$. Computing, we have that

$$\begin{aligned} f(z) &= \frac{1}{x+iy} \\ &= \frac{x-iy}{x^2+y^2} \\ &= \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2}\right) \\ &= u(x,y) + i v(x,y) \end{aligned}$$

(c) The domain of definition is $\mathbb{C} \setminus \{\pm i\}$. Computing, we have that

$$f(z)=\frac{z+i}{(z+i)(z-i)}$$
$$=\frac{1}{z-i}$$
$$=\frac{1}{x+iy-i}$$
$$=\frac{x-i(y-1)}{x^2+(1-y)^2}$$
$$=\frac{x}{x^2+(1-y)^2}+i\frac{(1-y)}{x^2+(1-y)^2}$$
$$=u(x,y)+iv(x,y)$$

(d) $f(z)=e^{3z}$
$$=e^{3(x+iy)}$$
$$=e^{3x}e^{3iy}$$
$$=e^{3x}\cos(3y)+ie^{3x}\sin(3y)$$
$$=u(x,y)+iv(x,y)$$

(e) The domain is $\mathbb{C}\setminus\{3\}$. Computing, we have that
$$f(z)=\frac{2(x+iy)^2+3}{|x-3+iy|}$$
$$=\frac{2(x^2-y^2)+3+2ixy}{\sqrt{(x-3)^2+y^2}}$$
$$=\frac{2(x^2-y^2)+3}{\sqrt{(x-3)^2+y^2}}+i\frac{2xy}{\sqrt{(x-3)^2+y^2}}$$
$$=u(x,y)+iv(x,y).$$

(f) The domain is $\mathbb{C}$. Computing, we have that
$$f(z)=e^{x+iy}+e^{-x-iy}$$
$$=e^x\cos(y)+ie^x\sin(y)+e^{-x}\cos(y)-ie^{-x}\sin(y)$$
$$=(e^x+e^{-x})\cos(y)+i(e^x-e^{-x})\sin(y)$$
$$=2\cosh(x)\cos(y)+2i\sinh(y)\sin(y)$$
$$=u(x,y)+iv(x,y).$$

$\blacksquare$

#4

Prove that the set of complex numbers $z_n = x_n + iy_n$ converges to $z_0 = x_0 + iy_0$ if and only if x_n converges to x_0 and y_n converges to y_0.

proof:

($\Rightarrow$) Suppose $z_n \to z_0$. Therefore, $|z_n - z_0| \to 0$. Consequently,

$$|x_n - x_0| \le \sqrt{(x_n-x_0)^2 + (y_n-y_0)^2} = |z_n - z_0|,$$

$$|y_n - y_0| \le \sqrt{(y_n-y_0)^2 + (x_n-x_0)^2} = |z_n - z_0|,$$

and thus by the squeeze theorem $x_n \to x_0$ and $y_n \to y_0$.

($\Leftarrow$) Suppose $x_n \to x_0$ and $y_n \to y_0$. Therefore, by the triangle inequality it follows that

$$\begin{aligned} |z_n - z_0| &= \sqrt{(x_n-x_0)^2 + (y_n-y_0)^2} \\ &\le \sqrt{(x_n-x_0)^2} + \sqrt{(y_n-y_0)^2} \\ &\le |x_n - x_0| + |y_n - y_0|. \end{aligned}$$

and thus by the squeeze theorem $|z_n - z_0| \to 0$.

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#5

Prove that the sequence of complex numbers satisfy $z_n \to z_0$ if and only if $\bar{z}_n \to \bar{z}_0$

proof:

Since $|z_n - z_0| = |\overline{z_n - z_0}| = |\bar{z}_n - \bar{z}_0|$ the result follows.

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#2.

For the complex function $f(z)=e^z$.

(a) Describe the domain of definition and range.

(c) Describe the image of the vertical line $Re(z)=1$.

(d) Describe the image of the horizontal line $Im(z)=\pi/4$.

(e) Describe the image of the infinite strip $0 \le Im(z) \le \pi/4$.

Solution:

(a) If we let $z=x+iy$, where $x,y \in \mathbb{R}$, it follows that

$$f(z)=e^x(\cos(y)+i\sin(y))$$
$$=e^x\cos(y)+ie^x\sin(y)$$
$$=u(x,y)+iv(x,y).$$

For a fixed value of x and varying y it follows that the image of $f(z)$ is a circle of radius e^x. Therefore, since $e^x > 0$ it follows that the range of f is $\mathbb{C}\setminus\{0\}$.

(c). For $Re(z)=1$ we obtain

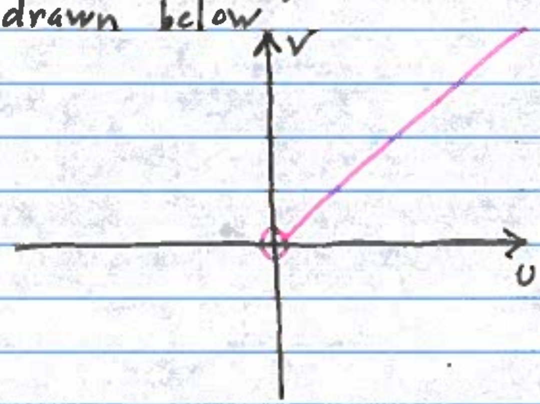
$$u(e,y)=e\cos(y)$$
$$v(e,y)=e\sin(y)$$

which is the parametric representation of a circle of radius e centered at the origin.

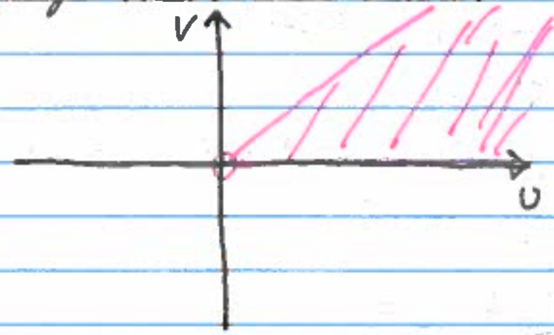
(d) For $Im(z)=\pi/4$ it follows that

$$u(x,\pi/4)=e^x/\sqrt{2}$$
$$v(x,\pi/4)=e^x/\sqrt{2}$$

and thus the image is the ray $v=u$ for $u>0$. This image is drawn below



(e) The image of the line $\text{Im}(z)=0$ is the positive real axis and thus the image of the region $0 \le \text{Im}(z) \le \pi/4$ is the wedge illustrated below.



#7

Compute the following limits justifying all steps or prove that that the limit does not exist.

(a) $\displaystyle\lim_{z\to 0} \frac{\text{Im}(z)}{z}$

(b) $\displaystyle\lim_{z\to 0} z e^{i\text{Re}(z)}$

(c) $\displaystyle\lim_{z\to 0} e^{1/z}$.

(d) $\displaystyle\lim_{z\to i} \frac{1}{z-i} - \frac{1}{z^2+1}$

Solution:

(a) Consider the sequences $z_n = 1/n$ and $w_n = i/n$ and let

$$f(z) = \frac{\text{Im}(z)}{z}.$$

Therefore,

$$f(z_n) = \frac{0}{1/n} = 0, \quad f(w_n) = \frac{1/n}{i/n} = -i$$

and thus

$$\lim_{n\to\infty} f(z_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} f(w_n) = -i$$

proving that $\displaystyle\lim_{z\to 0} f(z)$ does not exist.

(b). Let $z_n \in \mathbb{C}$ satisfy $z_n \to 0$. and $f(z) = z e^{i\,Re(z)}$. It follows that

$$|f(z_n) - 0| = |z_n e^{i\,Re(z_n)}|$$
$$= |z_n| \cdot |e^{i\,Re(z_n)}|$$
$$= |z_n|$$

and consequently $\lim_{n\to\infty} |f(z_n) - 0| = 0$ proving that

$$\lim_{n\to\infty} f(z_n) = 0$$

and thus $\lim_{z\to 0} f(z) = 0$.

(c) Consider the sequences $z_n = 1/n$, $w_n = i/n$. Consequently,

$$e^{1/z_n} = e^n \text{ and } e^{1/w_n} = e^{n/i} = e^{-in}.$$

Therefore,

$$\lim_{n\to\infty} e^{1/z_n} = \lim_{n\to\infty} e^n = \infty \quad \text{and} \quad |e^{1/w_n}| = 1$$

and consequently

$$\lim_{z\to 0} e^{1/z}$$

does not exist.

(d) Consider the sequence $z_n = i + 1/n$ and let

$$f(z) = \frac{1}{z-i} - \frac{1}{z^2+1}.$$

Therefore,

$$|f(z_n)| = \left| \frac{n}{i} - \frac{1}{-(1+1/n)^2+1} \right|$$
$$= \left| -in + \frac{1}{1+2/n+1/n^2-1} \right|$$
$$= \left| -in + \frac{n^2}{2n+1} \right|$$
$$= \sqrt{n^2 + \frac{n^4}{(2n+1)^2}}$$

which implies

$$\lim_{n\to\infty} |f(z_n)| = \infty$$

proving the limit does not exist.