MTH 317/617
Homework #4

Due Date: September 30, 2022

1 Problems for Everyone

1. Show that each of the following functions is nowhere differentiable using definition of the derivative, i.e. do not use the Cauchy Riemann equations.

   (a) $f(z) = \mathrm{Re}(z)$

   (b) $f(z) = \mathrm{Im}(z)$

   (c) $f(z) = |z|$

2. Let $P(z) = (z - z_1)(z - z_2)\cdots(z - z_n)$. Prove that

$$\frac{P'(z)}{P(z)} = \frac{1}{z - z_1} + \ldots + \frac{1}{z - z_n}$$

3. Verify that each given function u is harmonic (in the region where it is defined) and then find the harmonic conjugate of u.

   (a) $u = e^x \sin(y)$

   (b) $u = xy - x + y$

   (c) $u = \sin(x)\cosh(y)$

   (d) $u = \mathrm{Im}\exp(z^2)$

Homework #4

#1

Show that each of the functions is nowhere differentiable using the definition of the derivative.

(a) $f(z) = Re(z)$

(b) $f(z) = Im(z)$

(c) $f(z) = |z|$

Solution:

(a) Computing, we have that

$$\frac{f(z+1/n)-f(z)}{1/n} = \frac{Re(z+1/n)-Re(z)}{1/n} = \frac{1/n}{1/n} = 1,$$

$$\frac{f(z+i/n)-f(z)}{i/n} = \frac{Re(z+i/n)-Re(z)}{i/n} = \frac{0}{i/n} = 0.$$

Therefore,

$$\lim_{n\to\infty} \frac{f(z+1/n)-f(z)}{1/n} = 1 \neq 0 = \lim_{n\to\infty} \frac{f(z+i/n)-f(z)}{i/n}$$

and thus f is not differentiable.

(b) Computing, we have that

$$\frac{f(z+1/n)-f(z)}{1/n} = \frac{Im(z+1/n)-Im(z)}{1/n} = \frac{0}{1/n} = 0,$$

$$\frac{f(z+i/n)-f(z)}{i/n} = \frac{Im(z+i/n)-Im(z)}{i/n} = \frac{1/n}{i/n} = -i.$$

Therefore,

$$\lim_{n\to\infty} \frac{f(z+1/n)-f(z)}{1/n} = 0 \neq -i = \lim_{n\to\infty} \frac{f(z+i/n)-f(z)}{i/n}$$

and thus f is not differentiable.

(c) Computing, we have for $z=x+iy$ that

$$\frac{f(z+1/n)-f(z)}{1/n} = \frac{\sqrt{(x+1/n)^2+y^2}-\sqrt{x^2+y^2}}{1/n},$$

$$\frac{f(z+i/n)-f(z)}{i/n} = \frac{\sqrt{x^2+(y+1/n)^2}-\sqrt{x^2+y^2}}{i/n} = -i\left(\frac{\sqrt{x^2+(y+1/n)^2}-\sqrt{x^2+y^2}}{1/n}\right).$$

Therefore,

$$\lim_{n\to\infty}\frac{f(z+1/n)-f(z)}{1/n} = \frac{d}{dx}\sqrt{x^2+y^2} = \frac{x}{\sqrt{x^2+y^2}} \neq \frac{-iy}{\sqrt{x^2+y^2}} = -i\frac{d}{dy}\sqrt{x^2+y^2} = \lim_{n\to\infty}\frac{f(z+i/n)-f(z)}{i/n}$$

and thus f is not differentiable.

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#2.

Let $P(z)=(z-z_1)(z-z_2)\cdots(z-z_n)$. Prove that

$$\frac{P'(z)}{P(z)} = \frac{1}{z-z_1}+\dots+\frac{1}{z-z_n}.$$

proof 1:

Since $P(z)=(z-z_1)\cdots(z-z_n)=\prod_{i=1}^{n} z-z_i$ it follows from the product rule that

$$P'(z) = \sum_{j=1}^{n}\prod_{\substack{i=1\\ i\neq j}}^{n} z-z_i$$

and thus

$$\frac{P'(z)}{P(z)} = \frac{\sum_{j=1}^{n}\prod_{\substack{i=1\\ i\neq j}}^{n} z-z_i}{\prod_{k=1}^{n} z-z_k} = \sum_{j=1}^{n}\frac{1}{z-z_j}.$$

proof 2:

$$\frac{P'(z)}{P(z)} = \frac{d}{dz}\log(P(z)) = \frac{d}{dz}\log\left(\prod_{i=1}^{n}(z-z_i)\right) = \frac{d}{dz}\sum_{i=1}^{n}\log(z-z_i) = \sum_{i=1}^{n}\frac{1}{z-z}$$

#3

Verify that each given function u is harmonic and then find the harmonic conjugate of u.

(a) $u = e^x \sin(y)$

(b) $u = xy - x + y$

(c) $u = \sin(x)\cosh(y)$

(d) $u = \operatorname{Im}(\exp(z^2))$

Solution:

(a) $u_{xx} = e^x \sin(y)$ and $u_{yy} = -e^x \sin(y)$ and thus $u_{xx}+u_{yy}=0$.
Furthermore, $u_x = e^x \sin(y)$ and thus since $v_y = u_x$ it follows that $v = -e^x\cos(y) + \psi(x)$ for some function $\psi(x)$. Consequently, since $u_y = -v_x$ it follows that

$$e^x \cos(y) - \psi'(x) = e^x \cos(y)$$

and thus $\psi = 0$ works. Therefore,

$$v(x,y) = -e^x \cos(y).$$

(b) Calculating, we have that $u_{xx} = u_{yy} = 0$ and thus u is harmonic.
Furthermore, $u_x = y - 1 = v_y$ and thus $v = y^2/2 - y + \psi(x)$ for some function $\psi(x)$. Consequently, since $u_y = -v_x$ it follows that

$$x + 1 = -\psi'(x)$$

and thus $\psi(x) = \frac{-x^2}{2} - x$ works. Therefore,

$$v(x,y) = y^2/2 - y - x^2/2 - x.$$

(c) Computing we have that $u_{xx} = -\sin(x)\cosh(y)$ and $u_{yy} = \sin(x)\cosh(y)$ and thus $u_{xx}+u_{yy}=0$. Furthermore $u_x = \cos(x)\cosh(y) = v_y$ and thus $v = \cos(x)\sinh(y) + \psi(x)$ for some function $\psi(x)$. Therefore, since $u_y = -v_x$ it follows that

$$\sin(x)\sinh(y) = \sin(x)\sinh(y) - \psi'(x)$$

and thus $\psi = 0$ works. Therefore,

$$v(x,y) = \cos(x)\sinh(y).$$

(d). First, note that $\operatorname{Im}(\exp(z^2)) = \operatorname{Re}(i\exp(z^2))$ and $f(z) = i\exp(z^2)$ is analytic it follows that $u(x,y) = \operatorname{Im}(\exp(z^2))$ is harmonic. Therefore, the harmonic conjugate v is given by:

$$\begin{aligned}
v(x,y) &= \operatorname{Im}(-i\exp(z^2)) \\
&= \operatorname{Im}(-i\exp(x^2-y^2+2ixy)) \\
&= \operatorname{Im}(-i\exp(x^2-y^2)\exp(2ixy)) \\
&= \operatorname{Im}(-i\exp(x^2-y^2)(\cos(2xy)+i\sin(2xy)) \\
&= -e^{x^2-y^2}\underset{\cos}{\cancel{\sin}}(2xy).
\end{aligned}$$