MTH 317/617
Homework #5

Due Date: October 07, 2022

1 Problems for Everyone

1. Write the following polynomials in the Taylor form, centered at $z = 2$.
   (a) $p(z) = z^5 + 3z + 4$
   (b) $p(z) = z^{10}$
   (c) $p(z) = (z-1)(z-2)^3$.
2. If $p(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_0$ ($a_n \neq 0$), then its reverse polynomial $p^*(z)$ is given by
$$p^*(z) = \overline{a_n} + \overline{a_{n-1}}z + \ldots + \overline{a_0}z^n.$$
   (a) Show that $p^*(z) = z^n\overline{p(1/\overline{z})}$.
   (b) Show that if $p(z)$ has a zero at $z_0 \neq 0$ then $p^*(z)$ has a zero at $1/\overline{z_0}$.
   (c) Show that for $|z| = 1$, we have $|p(z)| = |p^*(z)|$.
3. Let $f(z)$ be the rational function defined by
$$f(z) = \frac{2z+i}{(z^2+z)(1-z)^2}.$$
   (a) Find all of the poles of this function and their multiplicities.
   (b) Find a partial fraction decomposition of this function.
   (c) If ζ is a pole of $f(z)$ then the coefficient of $\frac{1}{z-\zeta}$ in the partial fraction decomposition is called the residue of $f(z)$ at ζ and is denoted by $\mathrm{Res}(\zeta)$. Find the residues for all of the poles of this function.
4. Let $f : \mathbb{C} \mapsto \mathbb{C}$ be the complex cosine function $f(z) = \cos(z)$.
   (a) Use the Cauchy-Riemann equations to show that $\cos(z)$ is analytic function and prove that
$$\frac{df}{dz} = -\sin(z).$$
   (b) Compute the real and imaginary parts of the function $f(z^2)$.
   (c) Show that for $z \in \mathbb{C}$, $\arccos(z) = \cos^{-1}(z) = -i\mathrm{Log}(z + i\sqrt{1-z^2})$.
   (d) Show that for $z \in \mathbb{C}$, $\cosh(z) = \cos(iz)$.
5. **Logarithms**
   (a) Write $\log(1-i)$ in the form $x + iy$, where $x, y \in \mathbb{R}$.
   (b) Write $\mathrm{Log}(\sqrt{3}+i)$ in the form $x + iy$, where $x, y \in \mathbb{R}$.
   (c) Determine the domain of analyticity for $f(z) = \mathrm{Log}(4 + i - z)$.
   (d) Find all solutions $z \in \mathbb{C}$ to the equation $e^{2z} + e^z + 1 = 0$.

Homework #5

#1.

Write the following polynomials in Taylor Form, centered at $z=2$.

(a) $p(z)=z^5+3z+4$

(b) $p(z)=z^{10}$

(c) $p(z)=(z-1)(z-2)^3$

Solution:

(a) Computing, we have that

$$p'(z)=5z^4+3 \Rightarrow p'(2)=5\cdot 16+3$$

$$p''(z)=20z^3 \Rightarrow p''(2)=20\cdot 8$$

$$p'''(z)=60z^2 \Rightarrow p'''(2)=60\cdot 4$$

$$p^{iv}(z)=120z \Rightarrow p^{iv}(2)=120\cdot 2$$

$$p^{v}(z)=120 \Rightarrow p^{v}(2)=120$$

Therefore,

$$p(z)=p(2)+(5\cdot 16+3)(z-2)+\frac{20\cdot 8}{2}(z-2)^2+\frac{60\cdot 4}{3\cdot 2}(z-2)^3$$

$$+\frac{120\cdot 2}{4\cdot 3\cdot 2}(z-2)^4+\frac{120}{5\cdot 4\cdot 3\cdot 2}(z-2)^5$$

$$=42+8\overset{3}{\cancel{0}}(z-2)+80(z-2)^2+40(z-2)^3+10(z-2)^4+(z-2)^5. \blacksquare$$

#2.

If $p(z)=a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$ $(a_n \neq 0)$ then its reverse polynomial is given by

$$p^*(z) = \overline{a_n} + \overline{a_{n-1}}\, z + \dots + \overline{a_0}\, z^n.$$

(a) Show that $p^*(z) = z^n \overline{p(1/\overline{z})}$.

(b) Show that if $p(z)$ has a zero at $z_0 \neq 0$ then $p^*(z)$ has a zero at $1/\overline{z_0}$.

(c) Show that for $|z|=1$, we have $|p(z)| = |p^*(z)|$.

Solution:

(a) Computing we have that

$$\begin{aligned} p^*(z) &= \overline{a_n} + \overline{a_{n-1}}\, z + \dots + \overline{a_0}\, z^n \\ &= z^n\left(\overline{a_n}\, z^{-n} + \overline{a_{n-1}}\, z^{-n+1} + \dots + \overline{a_0}\right) \\ &= z^n\left(\left(\frac{\overline{a_n}}{\overline{\overline{z}^n}}\right) + \left(\frac{\overline{a_{n-1}}}{\overline{\overline{z}^{n-1}}}\right) + \dots + \overline{a_0}\right) \\ &= z^n\left(\overline{\frac{a_n}{\overline{z}^n} + \frac{a_{n-1}}{\overline{z}^{n-1}} + \dots + a_0}\right) \\ &= z^n\, \overline{p\left(\frac{1}{\overline{z}}\right)} \end{aligned}$$

(b) Computing, we have that

$$p^*(1/\overline{z_0}) = \frac{1}{\overline{z_0}^n}\, \overline{p(z_0)} = 0.$$

(c) If $|z|=1$ then $z\overline{z}=1$ and thus $z = 1/\overline{z}$. Consequently

$$|p^*(z)| = |z^n \overline{p(1/\overline{z})}| = |p(1/\overline{z})| = |p(z)|.$$

$\blacksquare$

#3.

Let $f(z)$ be the rational function defined by

$$f(z) = \frac{2z+i}{(z^2+z)(1-z)^2}.$$

(a) Find all of the poles of this function and their multiplicities.

(b) Find a partial fraction decomposition of this function.

(c) Find the residues for all of the poles of this function.

Solution:

(a) The poles are at $z=0, \overset{-1}{\pm i}$, and 1 with all of them having multiplicity 1 except $z=\overset{i}{1}$ which has multiplicity 2.

(b)
$$\frac{2z+i}{z(z^2+1)(1-z)^2} = \frac{A(z+i)(z-i)(1-z)^2 + Bz(z+i)(1-z)^2 + Cz(z-i)(1-z)^2 + Dz(z-i)(z+i)(1-z) + Ez(z-i)(z+i)}{z(z^2+1)(1-z)^2}$$

Taking cases:

1. $z=0 \Rightarrow i = A(i)(-i) = A$
   $\Rightarrow A = i.$
2. $z=i \Rightarrow 3i = B(i)(2i)(1-i)^2 = -2B(1-2i-1) = 4Bi$
   $\Rightarrow B = 3/4.$
3. $z=-i \Rightarrow -i = C(-i)(-2i)(1+i)^2 = -2C\,2i$
   $\Rightarrow C = 1/4.$
4. $z=1 \Rightarrow 2+i = E\cdot 1(1-i)(1+i) = 2E$
   $\Rightarrow E = 1 + i/2.$
5. $z=2i \Rightarrow 3i = i(3i)(i)(1-2i)^2 + \frac{3}{4}2i(3i)(1-2i)^2 + \frac{1}{4}2i(i)(1-2i)^2$
   $+ D\,2i\,i(1-2i) + (1+i/2)2i(i)(3i)$

Solve for D and the partial fraction decomposition is given by:

$$\frac{2z+i}{z(z^2+1)(1-z)^2} = \frac{i}{z} + \cancel{\frac{3/4}{(z-i)}} \cancel{+\frac{1/4}{(z+i)}} + \frac{\frac{1}{2}-\frac{i}{4}}{z+i} + \frac{\frac{1}{2}+\frac{3}{4}i}{1-z} + \frac{D}{1-z} + \frac{1+i/2}{(1-z)^2}.$$

(c) The residues for $z = 0, \cancel{\pm i}^{\,-1}$, and 1 are therefore:

$$\mathrm{Res}(0) = i$$

$$\cancel{\mathrm{Res}(i) = 3/4} \qquad \mathrm{Res}(-1) = \frac{1}{2} - \frac{i}{4}$$

$$\cancel{\mathrm{Res}(-i) = 1/4} \qquad \mathrm{Res}(1) = -\frac{1}{2} - \frac{3}{4} i$$

$$\cancel{\mathrm{Res}(0) = 1.}$$

#4

Let $f : \mathbb{C} \to \mathbb{C}$ be the complex cosine function $f(z) = \cos(z)$.

(a) Use the Cauchy Riemann equations to show that $\cos(z)$ is analytic and prove that

$$\frac{df}{dz} = -\sin(z).$$

(b) Compute the real and imaginary parts of $f(z^2)$.

(c) Show that for $z \in \mathbb{C}$, $\arccos(z) = \cos^{-1}(z) = -i\,\mathrm{Log}(z + i\sqrt{1 - z^2})$.

(d) Show that for $z \in \mathbb{C}$, $\cosh(z) = \cos(iz)$.

Solution:

(a) Splitting into real and imaginary parts we have that

$$f(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y).$$

Therefore,

$$\frac{\partial u}{\partial x} = -\sin(x)\cosh(y), \quad \frac{\partial v}{\partial x} = -\cos(x)\sinh(y)$$

$$\frac{\partial u}{\partial y} = \cos(x)\sinh(y), \quad \frac{\partial v}{\partial y} = -\sin(x)\cosh(y)$$

and thus the Cauchy-Riemann equations are satisfied. Therefore, f is analytic and

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -\sin(x)\cosh(y) - i\cos(x)\sinh(y) = -\sin(z).$$

(b) Computing, we have that

$$f(z^2) = f(x^2 - y^2 + 2ixy)$$
$$= \cos(x^2-y^2)\cosh(2xy) - i\sin(x^2-y^2)\sinh(2xy).$$

(c) Computing, we have that

$$\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$$

$$\Rightarrow 2e^{iz}\cos(z) = e^{2iz}+1$$
$$\Rightarrow e^{2iz} - 2e^{iz}\cos(z) + 1 = 0$$
$$\Rightarrow e^{iz} = \frac{2\cos(z) + \sqrt{4\cos^2(z)-4}}{2}$$
$$= \cos(z) + i\sqrt{1-\cos^2(z)}$$

Assuming the principal branch we have for some $n \in \mathbb{Z}$!

$$iz + 2n\pi i = \operatorname{Log}(\cos(z) + i\sqrt{1-\cos^2(z)})$$

thus

$$\operatorname{Arccos}(z) = -i\operatorname{Log}(z + i\sqrt{1-z^2}).$$

(d) Computing, we have that

$$\cos(iz) = \frac{e^{i(iz)}+e^{-i(iz)}}{2} = \frac{e^{-z}+e^{z}}{2} = \cosh(z).$$

■

#5

(a) Write $\log(1-i)$ in the form $x+iy$, where $x,y \in \mathbb{R}$.

(b) Write $\operatorname{Log}(\sqrt{3}+i)$ in the form $x+iy$, where $x,y \in \mathbb{R}$.

(c) Determine the domain of analyticity for $f(z) = \operatorname{Log}(4+i-z)$.

(d) Find all solutions $z \in \mathbb{C}$ to the equation $e^{2z}+e^{z}+1=0$.

Solution:

(a) Computing, we have that $1-i=\sqrt{2}e^{-i\pi/4+2n\pi i}$ and thus

$$\log(1-i)=\ln(\sqrt{2})+i(-\pi/4+2n\pi).$$

(b) Computing, we have that $\sqrt{3}+i=2e^{i\pi/6}$ and thus

$$\mathrm{Log}(\sqrt{3}+i)=\ln(2)+i\pi/6.$$

(c) Letting $z=x+iy$ we have that

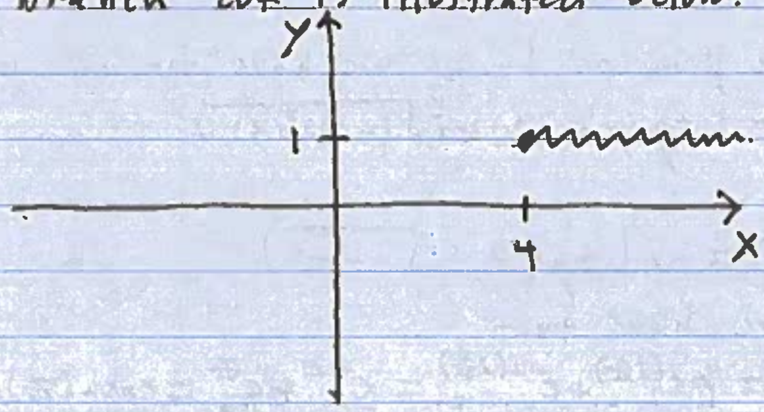
$$4+i-z=4-x+i(1-y)$$

and thus f will fail to be analytic when

$$4-x\le 0 \text{ and } 1-y=0$$

$$\Rightarrow x\ge 4 \quad \text{and} \quad y=1$$

The branch cut is illustrated below.



(d) Since $e^{2z}+e^{z}+1=0$ we have that

$$e^{z}=\frac{-1\pm\sqrt{1-4}}{2}$$

$$=\frac{-1\pm i\sqrt{3}}{2}$$

$$\Rightarrow z=\log\left(\frac{-1}{2}\pm i\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow z=\ln(1)\pm i\left(\frac{2\pi}{3}+2n\pi\right)$$

$$\Rightarrow z=\pm i\frac{2\pi}{3}+2n\pi i.$$