MTH 317/617

Homework #7

Due Date: October 21, 2022

1 Problems for Everyone

1. For each of the following curves give an admissible parametrization that is consistent with the indicated direction.

   (a) The line segment from $z = 1 + i$ to $z = -2 - 3i$.

   (b) The circle $|z - 2i| = 4$ transversed once in the clockwise direction starting from $z = 4 + 2i$.

   (c) The arc of the circle $|z| = R$ lying in the second quadrant, from $z = Ri$ to $z = -R$.

   (d) The segment of the parabola $y = x^2$ from the point $(1, 1)$ to the point $(3, 9)$.

2. Using an admissible parametrization, verify from the arclength integral that

   (a) The length of the line segment from z_1 to z_2 is $|z_1 - z_2|$.

   (b) The length of the circle $|z - z_0| = r$ is $2\pi r$.

3. In class we showed for $n \in \mathbb{Z}$ and C a circle of radius $r > 0$ centered at $z_0 \in \mathbb{C}$ that

$$\int_C (z - z_0)^n ds = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases}.$$

   Utilize this fact to evaluate the following contour integral

$$\int_C \left[\frac{6}{(z-i)^2} + \frac{2}{z-i} + 1 - 3(z-i)^2 \right] dz,$$

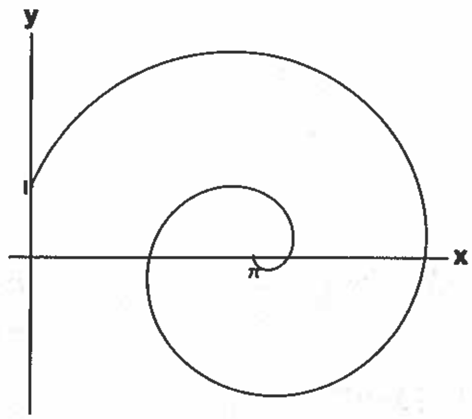
   where C is the circle $|z - i| = 4$ traversed once counterclockwise.

4. Let C be the perimeter of the square with vertices at the points $z = 0$, $z = 1$, $z = 1 + i$ and $z = i$ traversed once in that order.

   (a) Show by explicitly parametrizing C and computing the contour integral that $\int_C z^2 dz = 0$.

   (b) Show by explicitly parametrizing C and computing the contour integral that $\int_C \overline{z}^2 dz \neq 0$. Why does this result not violate the independence of path theorem?

5. The contour Γ drawn below starts at $z = \pi$ and ends at $z = i$.



Calculate the following integrals

(a) $\int_\Gamma (3z^2 - 5z + i)dz$

(b) $\int_\Gamma e^z dz$

(c) $\int_\Gamma \sin^2(z)\cos(z)dz$

(d) $\int_\Gamma e^z \cos(z)dz$

2 Graduate Problems

1. Let $z = z_1(t)$ be an admissible parametrization of the smooth curve γ. If $\phi(s)$, $c \le s \le d$ is a differentiable function satisfying $\phi'(s)$ is continuous, and $\phi(c) = a, \phi(d) = b$, then the function $z_2(s) = z_1(\phi(s))$, $c \le s \le d$ is also an admissible parametrization of γ. Verify that

$$\int_a^b |z_1'(t)|dt = \int_c^d |z_2'(s)|ds.$$

2. Let $f : \mathbb{C} \mapsto \mathbb{C}$ be an analytic function in all of $\mathbb{C}$, i.e. it is entire. Let γ be a smooth parametrization. Prove that

$$\left|\int_\gamma f(z)dz\right| \le \int_\gamma |f(z)|dz.$$

Hint: Start with the definition of the integral.

Homework #7

#1.

For each of the following curves give an admissible parametrization that is consistent with the indicated direction.

(a) The line segment from $z = 1+i$ to $z = -2-3i$.

(b) The circle $|z-2i| = 4$ transversed once in the clockwise direction starting from $z = 4+2i$.

(c). The arc of the circle $|z| = R$ lying in the second quadrant, from $z = Ri$ to $z = -R$.

(d). The segment of the parabola $y = x^2$ from the point $(1,1)$ to the point $(3,9)$.

Solution:

(a) For $t \in [0,1]$, $z(t) = (1-t)(1+i) + t(-2-3i)$.

(b) For $t \in [0,2\pi]$, $z(t) = 2i + 4e^{-it}$.

(c) For $t \in [\pi/2, \pi]$, $z(t) = Re^{it}$.

(d) For $t \in [1,3]$, $z(t) = t + it^2$.

#2.

Using an admissible parametrization, verify from the arclength integral that

(a) The length of the line segment from z_1 to z_2 is $|z_2 - z_1|$.

(b) The length of the circle $|z - z_0| = r$ is $2\pi r$.

Solution:

(a) For the parametrization $z(t) = (1-t)z_1 + tz_2$ with $t \in [0,1]$, we have that

$$\int_\gamma ds = \int_0^1 |z'(t)|\,dt = \int_0^1 |z_2 - z_1| = |z_2 - z_1|.$$

(b) For the parametrization $z(t) = z_0 + re^{it}$, with $t \in [0, 2\pi]$, we have that

$$\int_\gamma ds = \int_0^{2\pi} |z'(t)|\,dt = \int_0^{2\pi} |rie^{it}|\,dt = \int_0^{2\pi} r\,dt = 2\pi r.$$

#3.

Evaluate the following contour integral

$$\int_C \left[\frac{6}{(z-i)^2} + \frac{2}{z-i} + 1 - 3(z-i)^2\right] dz,$$

where C is the circle $|z-i| = 4$ transversed once counterclockwise.

Solution:

$$\int_C \left[\frac{6}{(z-i)^2} + \frac{2}{z-i} + 1 - 3(z-i)^2\right] dz = 6\int_C \frac{1}{(z-i)^2}\,dz + 2\int_C \frac{1}{z-i}\,dz + \int_C dz - 3\int_C (z-i)^2\,dz$$

$$= 6\cdot 0 + 2\cdot 2\pi i + 0 - 3\cdot 0$$

$$= 4\pi i.$$

#4.

Let C be the perimeter of the square with vertices at $z = 0, 1, 1+i, i$ transversed once in that order.

(a) Show by explicitly parametrizing C and computing the contour integral that $\int_C z^2\,dz = 0$.

(b) Show by explicity parametrizing C and computing the contour integral that $\int_C \bar{z}^2\,dz \neq 0$. Why does this result not violate the independence of path theorem?

Solution:

For both parts (a) and (b) $C = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ where $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are parametrized by

i.) $\gamma_1 : z_1(t) = t, \; t \in [0,1] \Rightarrow z_1'(t) = 1$

ii) $\gamma_2 : z_2(t) = 1 - t + t(1+i), \; t \in [0,1] \Rightarrow z_2'(t) = i$

iii) $\gamma_3 : z_3(t) = (1-t)(1+i) + ti, \; t \in [0,1] \Rightarrow z_3'(t) = -1$

iv) $\gamma_4 : z_4(t) = (1-t)i, \; t \in [0,1] \Rightarrow z_4'(t) = -i.$

(a) Computing, we have that

$$\int_C z^2 dz = \int_{\gamma_1} z^2 dz + \int_{\gamma_2} z^2 dz + \int_{\gamma_3} z^2 dz + \int_{\gamma_4} z^2 dz$$
$$= \int_0^1 t^2 dt + i\int_0^1 (1+it)^2 dt - \int_0^1 (1+i-t)^2 dt - i\int_0^1 (1-t)^2 dt$$
$$= \int_0^1 t^2 dt + i\int_0^1 (1+it)^2 dt - \int_0^1 (1+2(i-t)+(i-t)^2) dt - i\int_0^1 (1-t)^2 dt$$
$$= \int_0^1 t^2 dt + i\int_0^1 (1+2it-t^2) dt - \int_0^1 (1+2i-2t-1-2it+t^2) dt + i\int_0^1 (1-2t+t^2) dt$$
$$= \int_0^1 (t^2-2t+2t-t^2) dt + i\int_0^1 (1-t^2-2+2t+1-2t+t^2) dt$$
$$= 0$$

(b) Computing, we have that

$$\int_C \bar{z}^2 dz = \int_0^1 t^2 dt + i\int_0^1 (1-2it-t^2) dt - \int_0^1 (1-2i-2t-1+2it+t^2) dt + i\int_0^1 (1-2t+t^2) dt$$
$$= \int_0^1 (t^2+2t+2t-t^2) dt + i\int_0^1 (1-t^2+2 \overset{-}{+} 2t+1-2t+t^2) dt$$
$$= \int_0^1 4t \, dt + i\int_0^1 \overset{-4t}{4} \, dt$$
$$= 2 + \overset{2i}{4i} \neq 0.$$

This integral is not zero since $\bar{z}^2$ does not have an antiderivative.

#5.

The contour Γ starts at $z=\pi$ and ends at $z=i$, Calculate the following

(a) $\int_\Gamma (3z^2-5z+i)dz$

(b) $\int_\Gamma e^z dz$

(c) $\int_\Gamma \sin^2(z)\cos(z)dz$

(d) $\int_\Gamma e^z\cos(z)dz$

Solution: (The answers here have the bounds flipped.)

In each problem the function has an antiderivative and thus is path independent. Consequently,

(a) $\int_\Gamma (3z^2-5z+i)dz = z^3-\frac{5}{2}z^2+iz\Big|_i^\pi = \pi^3-\frac{5}{2}\pi^2+i\pi+i+\frac{5}{2}+1.$

(b) $\int_\Gamma e^z dz = e^z\Big|_i^\pi = e^\pi - e^i$

(c) $\int_\Gamma \sin^2(z)\cos(z)dz = \frac{1}{3}\sin^3(z)\Big|_i^\pi = -\frac{1}{3}\sin^3(i)$

(d) $\int_\Gamma e^z\cos(z)dz = \int_\Gamma e^z\left(\frac{e^{iz}+e^{-iz}}{2i}\right)dz = \frac{1}{2i}\int_\Gamma\left(e^{(1+i)z}+e^{(1-i)z}\right)dz$

$$\Rightarrow \int_\Gamma e^z\cos(z)dz = \frac{1}{2i}\left(\frac{1}{1+i}e^{(1+i)z}+\frac{1}{1-i}e^{(1-i)z}\right)\Big|_i^\pi$$

$$= \frac{1}{4i}\left(e^{(1+i)\pi}(1-i)+(1+i)e^{(1-i)\pi}-e^{(1+i)i}(1-i)-e^{(1-i)i}(1+i)\right)$$

$$= \frac{1}{4i}\left(e^\pi(i-1-1-i)-e^i\left(e^{-1}(1-i)+e^1(1+i)\right)\right)$$

$$= \frac{1}{4i}\left(-2e^\pi-e^i\left(e^{-1}(1-i)+e^1(1+i)\right)\right)$$

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