MTH 317/617
Homework #8

Due Date: November 04, 2022

1 Problems for Everyone

2 1. Determine the domain of analyticity for the following functions and explain why

$$\int_{|z|=2} f(z)dz = 0.$$

(a) $f(z) = \dfrac{z}{z^2+25}$

(b) $f(z) = \dfrac{\cos(z)}{z^2-6z+10}$

(c) $f(z) = \text{Log}(z+3)$

(d) $f(z) = \sec\left(\dfrac{z}{2}\right)$.

3 2. pg. 202, #13

2 3. Evaluate

$$\int_{\Gamma} \frac{z}{(z+2)(z-1)} dz,$$

where Γ is the circle $|z| = 4$ traversed once in the clockwise direction.

3 4. pg. 212, #3

Homework #8

#1

Determine the domain of analyticity for the following functions and explain why

$$\int_{|z|=2} f(z)\,dz = 0.$$

Solution:

In all of these problems $\int_{|z|=2} f(z)\,dz = 0$ since $f(z)$ will be analytic for $|z| \le 2$.

(a) $f(z) = \dfrac{z}{z^2+25}$ is analytic everywhere except at $z = \pm 5i$.

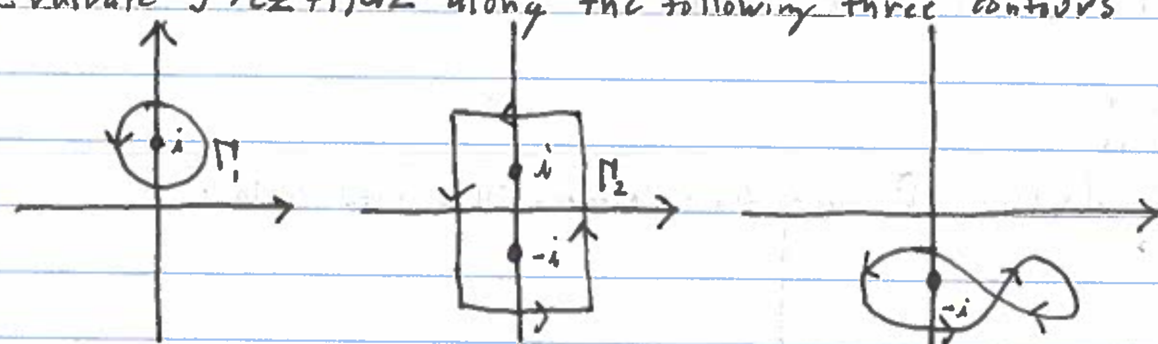
(b) $f(z) = \dfrac{\cos(z)}{z^2-6z+10}$ is analytic everywhere except $z = \dfrac{6 \pm \sqrt{36-40}}{2} = 3 \pm i$

(c) $f(z) = \mathrm{Log}(z+3)$ is analytic except where $\mathrm{Re}(z) \le -3$ and $\mathrm{Im}(z) = 0$.

(d) $f(z) = \sec(z/2)$ is analytic except where $z/2 = \pi/2 + n\pi \Rightarrow z = \pi + 2n\pi$ ∎

#2

Evaluate $\int 1/(z^2+1)\,dz$ along the following three contours



(a) (b)

Solution:

Since $1/z^2+1 = 1/(z+i)(z-i)$ we have that

(a) $\displaystyle\int_{\Gamma_1} \frac{1}{z^2+1}\,dz = \int_{\Gamma_1} \frac{1}{z-i}\cdot\frac{1}{z+i}\,dz = 2\pi i\cdot\frac{1}{z+i}\Big|_i = \pi$

(b) $\displaystyle\int_{\Gamma_2} \frac{1}{z^2+1}\,dz = \int_{|z-i|=1} \frac{1}{z^2+1}\,dz + \int_{|z+i|=1} \frac{1}{z^2+1}\,dz$

$$= \int_{|z-i|=1} \frac{1}{z-i}\,\frac{1}{z+i}\,dz + \int_{|z+i|=1} \frac{1}{z+i}\,\frac{1}{z-i}\,dz$$

$$= 2\pi i\,\frac{1}{z+i}\Big|_{z=i} + 2\pi i\,\frac{1}{z-i}\Big|_{z=-i}$$

$$= 0.$$

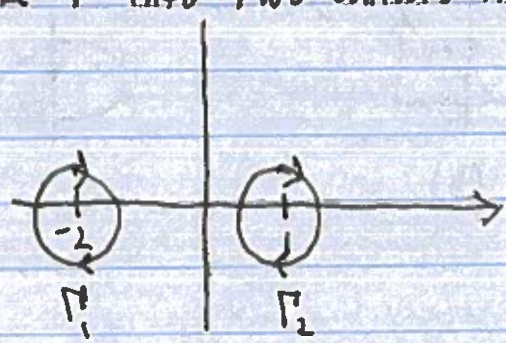
(c) $\displaystyle\int_{\Gamma_3} \frac{1}{z^2+1}\,dz = \int_{\Gamma_3} \frac{1}{z+i}\cdot\frac{1}{z-i}\,dz = 2\pi i\,\frac{1}{z-i}\Big|_{z=-i} = -\pi.$ ∎

#3.

Evaluate $\int_\Gamma z/(z+2)(z-1)\,dz$ where Γ is the circle $|z|=4$ transversed once in the clockwise direction.

Solution:

We deform Γ into two contours illustrated below:



Therefore, since each contour is oriented clockwise we have that

$$\int_{\Gamma} \frac{z}{(z+2)(z-1)}\,dz = \int_{\Gamma_1} \frac{1}{z+2}\cdot\frac{z}{z-1}\,dz + \int_{\Gamma_2} \frac{1}{z-1}\cdot\frac{z}{z+2}\,dz$$

$$= -2\pi i \cdot \frac{z}{z-1}\Big|_{z=-2} - 2\pi i\, \frac{z}{z+2}\Big|_{z=1}$$

$$= -\frac{4\pi i}{3} - \frac{2\pi i}{3}$$

$$= -2\pi i.$$

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Let C be the circle $|z|=2$ transversed once in the positive Sense. Compute each of the following integrals.

(a) $\displaystyle\int_C \frac{\sin(3z)}{z-\pi/2}\,dz$

(b) $\displaystyle\int_C \frac{ze^z}{2z-3}\,dz$

(c) $\displaystyle\int_C \frac{\cos(z)}{z^3+9z}\,dz$

Solution:

(a) $\displaystyle\int_C \frac{\sin(3z)}{z-\pi/2}\,dz = 2\pi i \sin(3z)\Big|_{\pi/2} = -2\pi i.$

(b) $\displaystyle\int_C \frac{ze^z}{2z-3}\,dz = \int_C \frac{ze^z}{2(z-3/2)}\,dz = \frac{2\pi i\, ze^z}{2}\Big|_{3/2} = \frac{3\pi i}{2} e^{3/2}.$

(c) $\displaystyle\int_C \frac{\cos(z)}{z(z^2+9)}\,dz = \int_C \frac{\cos(z)}{z(z+3i)(z-3i)}\,dz = \frac{\cos(z)\,2\pi i}{(z+3i)(z-3i)}\Big|_0 = \frac{2\pi i}{9}.$

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