MTH 317/617
Homework #9

Due Date: November 11, 2022

1 Problems for Everyone

1. pg. 202, #16.
2. pg. 203, #18.
3. pg. 212, #3 parts (d,e,f). I know I asked you to do this in the last homework, but **what I told students in office hours was wrong.** For part (d) rewrite the numerator as a polynomial centered at $z = i$ (we did this in a prior homework). For part (e) and (f) Taylor expand the numerators around the singularities $z = -1$ and $z = 0$ respectively. **Sorry about any confusion!**.
4. pg. 212, #4
5. pg. 212, #6, follow the problem's hint and Taylor expand the numerator for each integral.
6. For the following functions find the first five terms of the Taylor series about z_0 and determine the radius of convergence of the series

   (a) $\dfrac{1}{1+z}$, $z_0 = 0$.

   (b) e^{-z^2}, $z_0 = 0$.

   (c) $z^3 \sin(3z)$, $z_0 = 0$.

   (d) $z^3 \sin(3z)$, $z_0 = 0$.

   (e) $\dfrac{1+z}{1-z}$, $z_0 = i$.

   (f) $\dfrac{e^z}{3-2z}$, $z_0 = 0$.

   (g) $\dfrac{z}{(1-z)^2}$, $z_0 = 0$.

2

2

2

2

2

Homework #9

#2.

Let

$$I = \oint_{|z|=2} \frac{dz}{z^2(z-1)^3}.$$

Prove that $I = 0$.

~~proof~~

For every $R \geq 2$, $I = I(R)$, where

$$I(R) = \oint_{|z|=R} \frac{dz}{z^2(z-1)^3}$$

Since we can deform the contour to any circle of radius R containing the singularities. It follow from the triangle inequality and the fact we are integrating over a circle that

$$|I(R)| = \left| \oint_{|z|=R} \frac{dz}{z^2(z-1)^3} \right|$$

$$\leq \oint_{|z|=R} \frac{|dz|}{|z|^2|z-1|^3}$$

$$\leq \oint_{|z|=R} \frac{|dz|}{R^2(R-1)^3} = \frac{2\pi R}{R^2(R-1)^3} = \frac{2\pi}{R(R-1)^3}$$

Therefore, by the squeeze theorem

$$I = I(R) = \lim_{R\to\infty} I(R) = 0.$$

■

#3.

Let C be the circle $|z|=2$ transversed once in the positive sense. Compute each of the following integrals.

(d) $\displaystyle\int_C \frac{5z^2+2z+1}{(z-i)^3}\,dz$

(e) $\displaystyle\int_C \frac{e^{-z}}{(z+1)^2}\,dz$

(f) $\displaystyle\int_C \frac{\sin(z)}{z^2(z-4)}\,dz$

Solution:

(d) Letting $w=z-i$ we have $w+i=z$ and thus

$$\int_C \frac{5(w+i)^2+2(w+i)+1}{w^3}\,dw=\int_C \frac{5w^2+10wi-5+2w+2i+1}{w^3}\,dw$$

$$=\int_C \frac{5}{w}\,dw$$

$$=10\pi i.$$

(e). $$\int_C \frac{e^{-z}}{(z+1)^2}\,dz=\int_C \frac{e^{-(z+1)}e^{1}}{(z+1)^2}\,dz=\int_C \frac{e(1-(z+1)+\ldots)}{(z+1)^2}\,dz=-2\pi i e.$$

(f) $$\int_C \frac{\sin(z)}{z^2(z-4)}\,dz=\int_C \frac{z-z^3/3!+z^5/5!+\ldots}{z^2(z-4)}\,dz=\left.\frac{2\pi i}{z-4}\right|_{z=0}=-\frac{\pi i}{2}.$$

■

#4.

Compute $\int_C \frac{z+i}{z^3+2z^2}\,dz$, where C is

(a) $|z|=1$

(b) $|z+2-i|=2$

(c) $|z-2i|=1$

Solution:

$f(z) = \frac{z+i}{z^2(z+2)}$ has singularities at $z=0$ and $z=-2$.

(a)
$$f(z) = \frac{1}{z(z+2)} + \frac{i}{z^2(z+2)}$$
$$= \frac{1}{z(z+2)} + \frac{i}{2z^2(1+z/2)}$$
$$= \frac{1}{z(z+2)} + \frac{i}{2z^2}\left(1 - z/2 + z^2/4 + \ldots\right)$$

Therefore,
$$\int_C f(z)\,dz = \left.\frac{2\pi i}{z+2}\right|_{z=0} + 2\pi i\left(\frac{-i}{4}\right)$$
$$= \pi i + \frac{\pi}{2}.$$

(b) Since this contour does not contain 0 we have that
$$\int_C \frac{z+i}{z^2(z+2)}\,dz = 2\pi i \left.\frac{z+i}{z^2}\right|_{z=-2} = \frac{2\pi i(i-2)}{4}. \quad = -\frac{\pi}{2} - i\pi$$

(c) Since no singularities lie inside the contour,
$$\int_C \frac{z+i}{z^2(z+2)}\,dz = 0.$$

#5.

Evaluate
$$\int_\Gamma \frac{e^{iz}}{(1+z^2)^2}\,dz,$$
Where Γ is the circle $|z|=3$ transversed counterclockwise.

Solution:

Deforming Γ to two circles of radius 1, denoted Γ_1, Γ_2, centered at $\pm i$ we have that
$$\int_\Gamma \frac{e^{iz}}{(1+z^2)^2}\,dz = \int_{\Gamma_1} \frac{e^{iz}}{(1+z^2)^2}\,dz + \int_{\Gamma_2} \frac{e^{iz}}{(1+z^2)^2}\,dz$$
Therefore, letting $w_1 = z - i \Rightarrow w_1 + i = z$ we have that
$$\int_{\Gamma_1} \frac{e^{iz}}{(1+z^2)^2}\,dz = \int_{\Gamma_1} \frac{e^{i(w_1+i)}}{(1+(w_1+i)^2)^2}\,dw_1$$
$$= \int_{\Gamma_1} \frac{e^{-1}e^{iw_1}}{(2w_1 i + w_1^2)^2}\,dw_1$$
$$= \int_{\Gamma_1} \frac{e^{-1}(1 + iw_1 - w_1^2 + \dots)}{-4w_1^2(1 + w_1^2/2i)^2}\,dw_1$$
$$= \int_{\Gamma_1} \frac{e^{-1}(1 + iw_1 - w_1^2 + \dots)}{-4w_1^2}(1 - w_1^2/2i + \dots)(1 - w_1^2/2i + \dots)\,dw_1$$
$$= \int_{\Gamma_1} \frac{e^{-1}}{-4w_1^2} iw_1\,dw_1$$
$$= \frac{2\pi e^{-1}}{4}$$
$$= \frac{\pi e^{-1}}{2}.$$

Furthermore, letting $w_2 = z+i \Rightarrow w_2 - i = z$ we have that

$$\int_{\Gamma_2} \frac{e^{iz}}{(1+z^2)^2}\,dz = \int_{\Gamma_2} \frac{e^{i(w_2-i)}}{(1+(w_2-i)^2)^2}\,dw_2$$

$$= \int_{\Gamma_2} \frac{e\,e^{iw_2}}{(-2w_2 i + w_2^2)^2}\,dw_2$$

$$= \int_{\Gamma_2} \frac{e\,e^{iw_2}}{-4w_2^2(1 - w_2^2/2i)^2}\,dw_2$$

$$= \int_{\Gamma_2} \frac{e(1+iw_2 - \frac{1}{2}w_2^2 + \dots)}{-4w_2^2}(1+w_2^2/2i+\dots)(1+w_2^2/2i+\dots)\,dw_2$$

$$= \int_{\Gamma_2} \frac{ei}{-4w_2}\,dw_2$$

$$= \frac{2\pi e}{4}$$

$$= \frac{\pi}{2}e.$$

Consequently,

$$\int_{\Gamma} \frac{e^{iz}}{(1+z^2)^2}\,dz = \frac{\pi}{2}(e+e^{-1}) = \pi\cos(1).$$

$\blacksquare$

#6

For the following functions find the first five terms of the Taylor series about z_0 and determine the radius of convergence.

(a) $1/1+z$, $z_0 = 0$

(b) e^{-z^2}, $z_0 = 0$

(c) $z^3\sin(3z)$, $z_0 = 0$

(f) $e^z/3-2z$, $z_0 = 0$.

Solution:

(a) $\dfrac{1}{1+z} = 1 - z + z^2 - z^3 + z^4 + \cdots$

with radius of convergence $= 1$.

(b). $e^{-z^2} = 1 - z^2 + \dfrac{z^4}{2!} - \dfrac{z^6}{3!} + \dfrac{z^8}{4!} + \cdots$

With radius of convergence $= \infty$.

(c) $$z^3 \sin(3z) = z^3\left(3z - \frac{9z^3}{3!} + \frac{3^5 z^5}{5!} - \frac{3^7 z^7}{7!} + \frac{3^9 z^9}{9!} + \cdots\right)$$

$$= 3z^4 - \frac{3^3 z^6}{3!} + \frac{3^5 z^8}{5!} - \frac{3^7 z^{10}}{7!} + \frac{3^9 z^{12}}{9!} + \cdots$$

with radius of convergence $= \infty$.

(f) $$\frac{e^z}{3-2z} = \frac{e^z}{3(1-\frac{2}{3}z)} = \frac{(1 + z + z^2/2! + z^3/3! + z^4/4! + \cdots)}{3}\left(1 + \tfrac{2}{3}z + (\tfrac{2}{3})^2 z^2 + (\tfrac{2}{3})^3 z^3 + (\tfrac{2}{3})^4 z^4 + \cdots\right)$$

$$\Rightarrow \frac{e^z}{3-2z} = \frac{1}{3}\Big(1 + (\tfrac{2}{3}+1)z + (\tfrac{1}{2!} + \tfrac{2}{3} + (\tfrac{2}{3})^2)z^2 + ((\tfrac{2}{3})^3 + (\tfrac{2}{3})^2 + \tfrac{1}{2!}(\tfrac{2}{3}) + \tfrac{1}{3!})z^3 + ((\tfrac{2}{3})^4 + (\tfrac{2}{3})^3 + (\tfrac{2}{3})^2\tfrac{1}{2!} + (\tfrac{2}{3})\tfrac{1}{3!} + \tfrac{1}{4!})z^4 + \cdots$$

$$= \frac{1}{3} + \frac{5}{9}z + \frac{29}{54}z^2 + \frac{67}{182}z^3 + \frac{563}{1944}z^4 + \cdots$$

$\blacksquare$