MTH 317/617
Homework #1

Due Date: September 02, 2022

1 Problems for Everyone

1. Write the following complex expressions in the form $a + bi$ where a and b are real numbers.

   (a) $(-1 + i)^2$,

   (b) $\dfrac{(8 + 2i) - (1 - i)}{(2 + i)^2}$,

   (c) $\dfrac{2 + 3i}{1 + 2i} - \dfrac{8 + i}{6 - i}$,

   (d) $\left(\dfrac{2 + i}{6i - (1 - 2i)}\right)^2$.

2. Solve the following equations for z. Express your answer in the form $z = a + bi$ where a and b are real numbers.

   (a) $iz = 4 - zi$,

   (b) $\dfrac{z}{1 - z} = 1 - 5i$,

   (c) $(2 - i)z + 8z^2 = 0$,

   (d) $z^2 + 16 = 0$.

3. Let $z \in \mathbb{C}$ and assume $z \neq 0$. Prove the following:

   (a) $|\text{Re}(z)| \leq |z|$ and $|\text{Im}(z)| \leq |z|$,

   (b) $\text{Re}(z) = \dfrac{z + \overline{z}}{2}$ and $\text{Im}(z) = \dfrac{z - \overline{z}}{2i}$,

   (c) If k is an integer then $(\overline{z})^k = \overline{(z^k)}$,

   (d) If $|z| = 1$ and $z \neq 1$, then $\text{Re}\left((1 - z)^{-1}\right) = \dfrac{1}{2}$.

4. Describe the set of point $z \in \mathbb{C}$ that satisfy each of the following.

   (a) $|z - 1 + 1| = 3$,

   (b) $|z - 1| = |z + 1|$,

   (c) $|z| = \text{Re}(z) + 2$,

   (d) $2 < |z| < 6$.

5. Find the argument of the following complex numbers and write each in the polar form $z = r(\cos(\theta) + i\sin(\theta))$.

   (a) $-\frac{1}{2}$,

   (b) $-3 + 3i$,

   (c) $-\pi i$,

   (d) $-2\sqrt{3} - 2i$.

6. Write the given complex number in the form $a + bi$, where $a, b \in \mathbb{R}$.

   (a) $e^{-i\frac{\pi}{2}}$,

   (b) $\frac{e^{1+3\pi i}}{e^{-1+\frac{\pi i}{2}}}$,

   (c) $\frac{e^{3i} - e^{-3i}}{2i}$,

   (d) e^{e^i}.

2 Graduate Problems

1. Let B be an $m \times n$ matrix with complex valued entries. By $B^\dagger$ we denote the $n \times m$ matrix obtained by forming the transpose of B followed by taking the conjugate of each entry. Let A be an $n \times n$ matrix with complex entries. Prove that if $\mathbf{u}^\dagger A\mathbf{u} = 0$ for all $n \times 1$ column vectors $\mathbf{u}$ with complex entries, then A is the 0 matrix.