MTH 317/617
Homework #2

Due Date: September 09, 2022

1 Problems for Everyone

1. Show that the function $z(t) = e^{it}$, $0 \leq t \leq 2\pi$, is the parametric representation of a the unit circle in the complex plane traversed in the counterclockwise direction.

2. Sketch the following curves in the complex plane:
   (a) $z(t) = 3e^{it}, 0 \leq t \leq 2\pi$.
   (b) $z(t) = 2e^{-it}, 0 \leq t \leq \pi$.
   (c) $z(t) = e^{-(1+i)t}, 0 \leq t \leq 2\pi$.
   (d) $z(t) = e^{(1+i)t}, 0 \leq t \leq 2\pi$.

3. Find all the values of the following. Express your answers in the form $z = a+ib$ where $a, b \in \mathbb{R}$.
   (a) $(-16)^{1/4}$.
   (b) $i^{1/4}$.
   (c) $(1 - \sqrt{3}i)^{1/3}$.
   (d) $\left(\dfrac{2i}{1+i}\right)^{1/6}$.
   (e) i^i.

4. Let $a_0, \ldots, a_n \in \mathbb{R}$ and assume $z^* \in \mathbb{C}$ is a root of the polynomial

$$p(z) = a_0 + a_1 z + \ldots a_n z^n.$$

   (a) Prove that for all $z \in \mathbb{C}$, $p(\overline{z}) = \overline{p(z)}$.
   (b) Prove that $\overline{z^*}$ is also a root of the polynomial.

5. Sketch each of the following sets and determine which of the sets are *domains:*
   (a) $\{z \in \mathbb{C} : |z - 1 + i| \leq 3\}$
   (b) $\{z \in \mathbb{C} : 0 < |z - 2| < 3\}$
   (c) $\{z \in \mathbb{C} : |z| \geq 2\}$
   (d) $\{z \in \mathbb{C} : -1 < \text{Im}(z) \leq 1\}$
   (e) $\{z \in \mathbb{C} : (\text{Re}(z))^2 > 1\}$

6. For each of the following points in $\mathbb{C}$, determine its stereographic projection on the Riemann sphere
   (a) $z = i$
   (b) $z = 6 - 8i$
   (c) $z = -\frac{3}{10} + \frac{2}{5}i$.
7. Describe the projections on the Riemann sphere of the following sets in the complex plane $\mathbb{C}$:
   (a) The right half plane $\{z : \text{Re}(z) > 0\}$
   (b) The disk $\{z : |z| < 1/2\}$
   (c) The annulus $\{z : 1 < |z| < 2\}$
   (d) The set $\{z : |z| > 3\}$
   (e) The line $y = x$ (including the point at ∞).

2 Graduate Problems

1. The following problems are related to the topology of $\mathbb{C}$.
   (a) If S_i is a countably infinite collection of open subsets of $\mathbb{C}$ indexed by $i \in \mathbb{N}$, prove that $\bigcup_{i=1}^{\infty} S_i$ is open.
   (b) If S_i is a finite collection of open subsets of $\mathbb{C}$ indexed by $i \in \{1, \ldots, n\}$, prove that $\bigcap_{i=1}^{n} S_i$ is open.
   (c) If S_i is a countably infinite collection of open subsets of $\mathbb{C}$ indexed by $i \in \mathbb{N}$, then $\bigcap_{i=1}^{\infty} S_i$ is not necessarily open. Show this by constructing an explicit example.
   (d) Prove that if S and T are domains in $\mathbb{C}$ with at least one point in common, then $S \cup T$ is a domain.
   (e) If S and T are domains, is $S \cap T$ a domain? If so, prove it. It not, draw a counterexample.