MTH 317/617
Homework #5

Due Date: October 07, 2022

1 Problems for Everyone

1. Write the following polynomials in the Taylor form, centered at $z = 2$.
   (a) $p(z) = z^5 + 3z + 4$
   (b) $p(z) = z^{10}$
   (c) $p(z) = (z-1)(z-2)^3$.
2. If $p(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_0$ $(a_n \neq 0)$, then its reverse polynomial $p^*(z)$ is given by
$$p^*(z) = \overline{a_n} + \overline{a_{n-1}}z + \ldots + \overline{a_0}z^n.$$
   (a) Show that $p^*(z) = z^n\overline{p(1/\overline{z})}$.
   (b) Show that if $p(z)$ has a zero at $z_0 \neq 0$ then $p^*(z)$ has a zero at $1/\overline{z_0}$.
   (c) Show that for $|z| = 1$, we have $|p(z)| = |p^*(z)|$.
3. Let $f(z)$ be the rational function defined by
$$f(z) = \frac{2z+i}{(z^2+z)(1-z)^2}.$$
   (a) Find all of the poles of this function and their multiplicities.
   (b) Find a partial fraction decomposition of this function.
   (c) If ζ is a pole of $f(z)$ then the coefficient of $\frac{1}{z-\zeta}$ in the partial fraction decomposition is called the residue of $f(z)$ at ζ and is denoted by $\mathrm{Res}(\zeta)$. Find the residues for all of the poles of this function.
4. Let $f : \mathbb{C} \mapsto \mathbb{C}$ be the complex cosine function $f(z) = \cos(z)$.
   (a) Use the Cauchy-Riemann equations to show that $\cos(z)$ is analytic function and prove that
$$\frac{df}{dz} = -\sin(z).$$
   (b) Compute the real and imaginary parts of the function $f(z^2)$.
   (c) Show that for $z \in \mathbb{C}$, $\arccos(z) = \cos^{-1}(z) = -i\mathrm{Log}(z + i\sqrt{1-z^2})$.
   (d) Show that for $z \in \mathbb{C}, \cosh(z) = \cos(iz)$.
5. **Logarithms**
   (a) Write $\log(1-i)$ in the form $x + iy$, where $x, y \in \mathbb{R}$.
   (b) Write $\mathrm{Log}(\sqrt{3} + i)$ in the form $x + iy$, where $x, y \in \mathbb{R}$.
   (c) Determine the domain of analyticity for $f(z) = \mathrm{Log}(4 + i - z)$.
   (d) Find all solutions $z \in \mathbb{C}$ to the equation $e^{2z} + e^z + 1 = 0$.