

MTH 317/617

Homework #7

Due Date: October 21, 2022

1 Problems for Everyone

- For each of the following curves give an admissible parametrization that is consistent with the indicated direction.
 - The line segment from $z = 1 + i$ to $z = -2 - 3i$.
 - The circle $|z - 2i| = 4$ transversed once in the clockwise direction starting from $z = 4 + 2i$.
 - The arc of the circle $|z| = R$ lying in the second quadrant, from $z = Ri$ to $z = -R$.
 - The segment of the parabola $y = x^2$ from the point $(1, 1)$ to the point $(3, 9)$.
- Using an admissible parametrization, verify from the arclength integral that
 - The length of the line segment from z_1 to z_2 is $|z_1 - z_2|$.
 - The length of the circle $|z - z_0| = r$ is $2\pi r$.
- In class we showed for $n \in \mathbb{Z}$ and C a circle of radius $r > 0$ centered at $z_0 \in \mathbb{C}$ that

$$\int_C (z - z_0)^n ds = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases}.$$

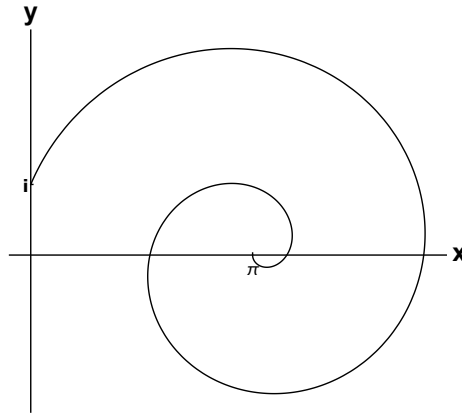
Utilize this fact to evaluate the following contour integral

$$\int_C \left[\frac{6}{(z-i)^2} + \frac{2}{z-i} + 1 - 3(z-i)^2 \right] dz,$$

where C is the circle $|z - i| = 4$ traversed once counterclockwise.

- Let C be the perimeter of the square with vertices at the points $z = 0$, $z = 1$, $z = 1 + i$ and $z = i$ traversed once in that order.
 - Show by explicitly parametrizing C and computing the contour integral that $\int_C z^2 dz = 0$.
 - Show by explicitly parametrizing C and computing the contour integral that $\int_C \bar{z}^2 dz \neq 0$. Why does this result not violate the independence of path theorem?

5. The contour Γ drawn below starts at $z = \pi$ and ends at $z = i$.



Calculate the following integrals

- (a) $\int_{\Gamma} (3z^2 - 5z + i) dz$
 (b) $\int_{\Gamma} e^z dz$
 (c) $\int_{\Gamma} \sin^2(z) \cos(z) dz$
 (d) $\int_{\Gamma} e^z \cos(z) dz$

2 Graduate Problems

1. Let $z = z_1(t)$ be an admissible parametrization of the smooth curve γ . If $\phi(s)$, $c \leq s \leq d$ is a differentiable function satisfying $\phi'(s)$ is continuous, and $\phi(c) = a, \phi(d) = b$, then the function $z_2(s) = z_1(\phi(s))$, $c \leq s \leq d$ is also an admissible parametrization of γ . Verify that

$$\int_a^b |z_1'(t)| dt = \int_c^d |z_2'(s)| ds.$$

2. Let $f : \mathbb{C} \mapsto \mathbb{C}$ be an analytic function in all of \mathbb{C} , i.e. it is entire. Let γ be a smooth parametrization. Prove that

$$\left| \int_{\gamma} f(z) dz \right| \leq \int_{\gamma} |f(z)| dz.$$

Hint: Start with the definition of the integral.