Lecture 10: Analyticity and the Cauchy Riemann Equations.

Definition - Let $f: \mathbb{C} \to \mathbb{C}$. Then,

$$\frac{df}{dz}(z_0) = f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

A function that is differentiable at z_0 is called analytic at z_0.

Example:

If $f(z) = z^2$, then $f'(z) = 2z$.

proof

Let Δz_n be a sequence satisfying $\Delta z_n \neq 0$ and $\Delta z_n \to 0$. Then,

$$\left| \frac{f(z+\Delta z) - f(z)}{\Delta z_n} - 2z \right| = \left| \frac{z^2 + 2z\Delta z_n + \Delta z_n^2 - z^2 - 2z\Delta z_n}{\Delta z_n} \right|$$

$$= \left| \frac{\Delta z_n^2}{\Delta z_n} \right|$$

$$= |\Delta z_n|$$

Since $\lim_{n\to\infty} \Delta z_n = 0$ it follows that $\lim_{n\to\infty} |\Delta z_n| = 0$ and thus

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} - 2z = 0$$

and therefore

$$f'(z) = 2z.$$

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Example:

Let $f(z) = \mathrm{Re}(z)$. Then, $f(z)$ is not differentiable.

proof:

Let $\Delta z_n = 1/n$. Then,

$$\lim_{n\to\infty} \frac{f(z+\Delta z_n) - f(z)}{\Delta z_n} = \lim_{n\to\infty} 1 = 1.$$

Let $\Delta w_n = i/n$. Then,

$$\lim_{n\to\infty} \frac{f(z+\Delta z_n) - f(z)}{\Delta z} = \lim_{n\to 0} \frac{0}{i/n} = 0.$$

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Example:

$f(z) = \bar{z}$ is nowhere differentiable.

proof

Let $\Delta z_n = i/n$, $\Delta w_n = 1/n$. Then

$$\frac{f(z+\Delta z_n) - f(z)}{\Delta z_n} = \frac{-i/n}{i/n} = -1$$

$$\frac{f(z+\Delta w_n) - f(z)}{\Delta w_n} = \frac{1/n}{1/n} = 1.$$

Cauchy-Riemann Equations:

We know that

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

Let $f(x,y) = u(x,y) + i v(x,y)$. Consider

$$\Delta z_n = 1/n$$
$$\Delta w_n = i/n$$

If f is analytic then

- $$\lim_{n\to\infty} \frac{f(z+\Delta z_n) - f(z)}{\Delta z_n} = \lim_{n\to\infty} \frac{u(x+1/n, y) - u(x,y)}{1/n} + i \lim_{n\to\infty} \frac{v(x+1/n,y) - v(x,y)}{1/n}$$
$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}.$$

- $$\lim_{n\to\infty} \frac{f(z+\Delta w_n) - f(z)}{\Delta w_n} = \lim_{n\to\infty} \frac{u(x,y+1/n) - u(x,y)}{i/n} + i \lim_{n\to\infty} \frac{v(x,y+1/n) - v(x,y)}{1/n}$$
$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$
$$= \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

Therefore,

- $$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$
- $$\underbrace{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}_{\text{Cauchy-Riemann Equations}}$$

Theorem - If

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

and $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous then f is analytic.

Example:

If $f(z) = e^z$, then what is $f'(z)$?

$$f(z) = e^x(\cos(y) + i\sin(y))$$

$$u(x,y) = e^x\cos(y), \quad v(x,y) = e^x\sin(y)$$

$$\Rightarrow \frac{\partial u}{\partial x} = e^x\cos(y), \quad \frac{\partial u}{\partial y} = -e^x\sin(y), \quad \frac{\partial v}{\partial x} = e^x\sin(y), \quad \frac{\partial v}{\partial y} = e^x\cos(y)$$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, f is differentiable and

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x\cos(y) + ie^x\sin(y) = e^x(\cos(y) + i\sin(y)) = e^z.$$

Consequence:

If f, g are analytic:

1. $(f+g)' = f' + g'$
2. $(fg)' = f'g + fg'$
3. $\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$
4. $(f(g))' = f'(g) \cdot g'$.

Theorem: $\frac{dz^m}{dz} = mz^{m-1}$

proof:

Let $\Delta z_n \in \mathbb{C}$ satisfying $\Delta z_n \to 0$ and $\Delta z_n \neq 0$. Then,

$$\left| \frac{(z+\Delta z_n)^m - z^m}{\Delta z_n} - mz^{m-1} \right| = \left| \frac{z^m + mz^{m-1}\Delta z_n + c_1 z^{m-2}\Delta z_n^2 + \dots + \Delta z^m - z^m}{\Delta z_n} - mz^{m-1} \right|$$

$$= |c_1 z^{m-2}\Delta z_n + \dots + \Delta z_n^{m-1}|$$

$$\leq |c_1| \cdot |z^{m-2}| \cdot |\Delta z_n| + \dots + |\Delta z_n^{m-1}|$$

By the squeeze theorem

$$\lim_{\Delta z_n \to 0} \left| \frac{(z+z_n)^m - z^m}{\Delta z_n} - mz^{m-1} \right| = 0$$

$$\Rightarrow \frac{dz^m}{dz} = mz^{m-1}$$