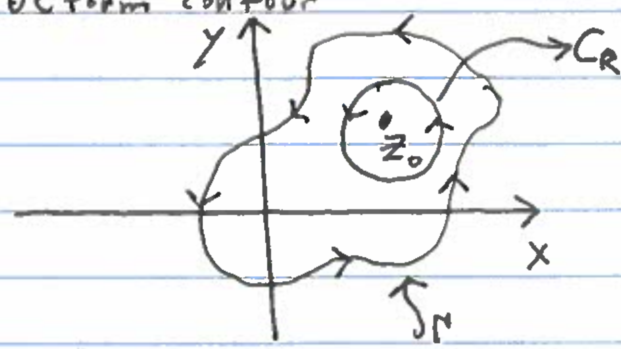
Lecture 19: Cauchy Integral Theorem and its Consequences

Theorem - Let Γ be a closed contour. If f is analytic in a simply connected domain D containing Γ and z_0 is any point inside Γ, then

$$f(z_0) = \frac{1}{2\pi i}\int_D \frac{f(z)}{z-z_0}\,ds \Rightarrow 2\pi i f(z_0) = \int_D \frac{f(z)}{z-z_0}.$$

proof:

Deform contour



$$\Rightarrow \int_\Gamma \frac{f(z)}{z-z_0}\,dz = \int_{C_R} \frac{f(z)}{z-z_0}\,dz,$$

$$= \int_{C_R} \frac{f(z_0)}{z-z_0}\,dz + \int_{C_R} \frac{f(z)-f(z_0)}{z-z_0}\,dz,$$

$$= 2\pi i f(z_0) + \underbrace{\int_{C_R} \frac{f(z)-f(z_0)}{z-z_0}\,dz}_{\text{function of } R}.$$

$$\left|\int_{C_R} \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| \le \int_{C_R} \frac{|f(z)-f(z_0)||dz|}{|z-z_0|}$$

$$= \int_{C_R} \frac{|f(z)-f(z_0)||dz|}{R}$$

Since f is continuous it follows that $|f(z)-f(z_0)|$ is continuous and thus by the extreme value theorem there exists $M(R)$

such that $M(R) = \max_{z \in C_R} |f(z) - f(z_0)|$ and $\lim_{R \to 0} M(R) = 0$. Therefore, since $z(t) = z_0 + Re^{it}$ parametrizes C_R it follows that

$$\left| \int_{C_R} \frac{f(z) - f(z_0)}{z - z_0} dz \right| \leq \int_{C_R} \frac{M(R)}{R} |dz| = \int_0^{2\pi} \frac{M(R)}{R} R\, dt = 2\pi M(R)$$

and thus by the squeeze theorem

$$\lim_{R \to 0} \int_{C_R} \frac{f(z) - f(z_0)}{z - z_0} dz = 0.$$

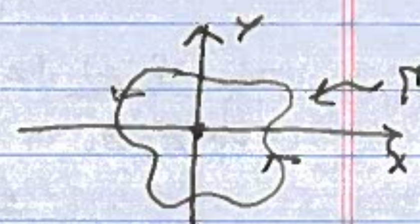
Therefore,

$$\lim_{R \to 0} \int_{\Gamma} \frac{f(z)}{z - z_0} dz = \lim_{R \to 0} \int_{C_R} \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0) + \lim_{R \to 0} \int_{C_R} \frac{f(z) - f(z_0)}{z - z_0} dz$$

$$\Rightarrow \int_{\Gamma} \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0).$$

Example:

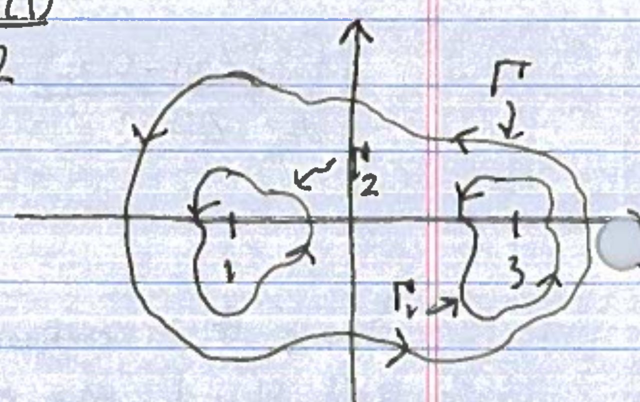
1. $$\int_{\Gamma} \frac{e^z + \sin(z)}{z} dz = 2\pi i (e^0 + \sin(0)) = 2\pi i$$



2. $$\int_{\Gamma} \frac{\cos(z)}{(z-3)(z-1)} dz = \int_{\Gamma_1} \frac{\cos(z)}{z-1} \cdot \frac{1}{z-3} dz + \int_{\Gamma_2} \frac{\cos(z)}{z-3} \frac{1}{z-1} dz$$

$$= 2\pi i \frac{\cos(3)}{2} + 2\pi i \frac{\cos(1)}{-2}$$

$$= \pi i (\cos(3) - \cos(1))$$



Theorem - Let g be an analytic function on a simply connected domain D. Then,

$$g(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{g(\zeta)}{\zeta - z} \, d\zeta$$

and

$$g^{(n)}(z) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{g(\zeta)}{(\zeta - z)^n} \, d\zeta$$

* Consequence, if g is analytic then all of its derivatives exist and are analytic!!