Lecture 21: Taylor Series

Let $f: \mathbb{C} \to \mathbb{C}$ be analytic in a domain D. The series:

$$f(z_0) + f'(z_0)(z - z_0) + \frac{1}{2} f''(z_0)(z - z_0)^2 + \dots$$

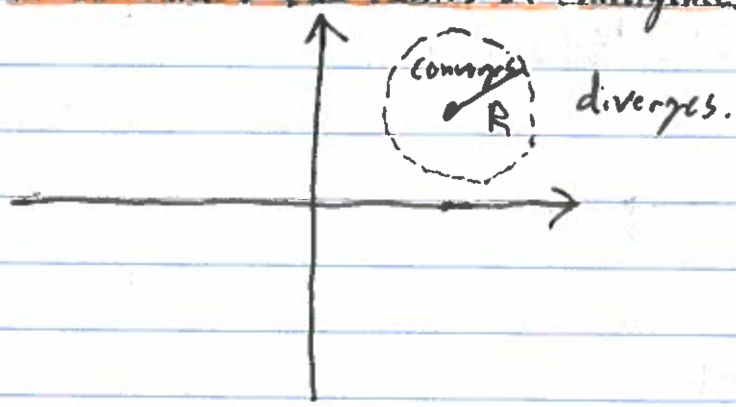
is called the Taylor series of $f(z)$ about z it is also a power series.

Theorem: For any power series $\sum_{j=0}^{\infty} a_j (z - z_0)^j$ there exists $R \in [0, \infty]$ such that:

i. The series converges for $|z - z_0| < R$

ii. The series diverges for $|z - z_0| > R$

R is called the radius of convergence.



Why does this work??

* Ratio test:

$$\lim_{j \to \infty} \left| \frac{a_{j+1}(z - z_0)^{j+1}}{a_j (z - z_0)^j} \right| = \lim_{j \to \infty} \left| \frac{a_{j+1}}{a_j} \right| \cdot |z - z_0| = L \cdot |z - z_0| < 1$$

$$\Rightarrow |z - z_0| < 1/L$$

Converges if $|z - z_0| < 1/L$ and diverges if $|z - z_0| > 1/L$.

Theorem - A power series sums to a function that is analytic inside its radius of convergence.

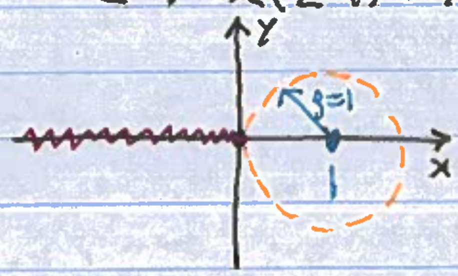
Examples:

Compute the Taylor series and the radius of convergence for the following functions.

1. $\mathrm{Log}(z)$, $z_0 = 1$

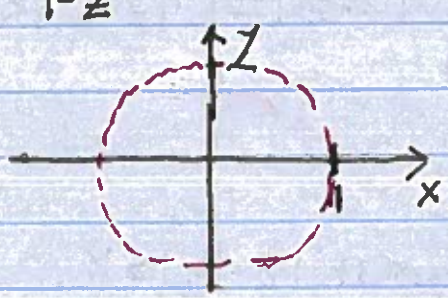
$$\frac{d}{dz}\mathrm{Log}(z) = 1/z$$
$$\frac{d}{dz} 1/z = -1/z^2$$
$$\vdots$$

$$\mathrm{Log}(z) = \mathrm{Log}(z_0) + 1/z_0 (z - z_0) - 1/2z_0^2 (z - z_0)^2 + \ldots$$
$$= z - 1 - 1/2 (z-1)^2 + 1/3 (z-1)^3 + \ldots$$



Radius of convergence $= 1$.

2. $\dfrac{1}{1-z}$, $z_0 = 1$



$$\frac{1}{1-z} = 1 + z + z^2 + \ldots$$

Radius of convergence $= 1$.

3. $\tan(z)$, $z_0 = 0$

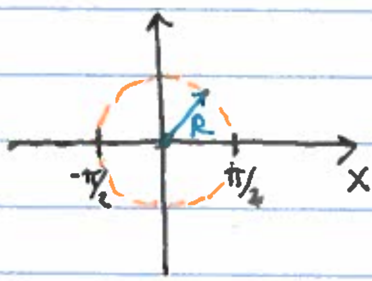
$$\tan(z) = \frac{\sin(z)}{\cos(z)} = \frac{z - z^3/3! + z^5/5! + \ldots}{1 - z^2/2! + z^4/4! + \ldots}$$

$$= \frac{z - z^3/3! + z^5/5! + \ldots}{1 - w}$$

$$= (z - z^3/3! + z^5/5! + \ldots)(1 + w + w^2 + w^3 + \ldots)$$

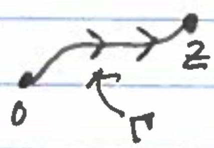
$$= (z - z^3/3! + z^5/5! + \ldots)(1 + (z^2/2! - z^4/4! + \ldots) + (z^2/2! - z^4/4! + \ldots)^2 + \ldots$$

$$= z - \frac{z^3}{3!} + \left(\frac{-1}{4!} - \frac{1}{2!3!} + \frac{1}{5!}\right) z^5 + \ldots$$



Radius of convergence $= \pi/2$.

4. $F(z) = \int_\Gamma e^{-w^2} dw$, $w \in \mathbb{C}$



$$F(z) = \int_\Gamma \left(1 - w^2 + \frac{1}{2!} w^4 - \frac{1}{3!} w^6 + \ldots\right) dw$$

$$= z - \frac{1}{3} z^3 + \frac{1}{2! \cdot 5} z^5 - \frac{1}{3! \cdot 7} z^7$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} z^{2n+1}.$$

5. $$\frac{1}{(1-z)^3} = \frac{1}{2} \frac{d^2}{dz^2} \frac{1}{(1-z)} = \frac{1}{2} \frac{d^2}{dz^2} (1 + z + z^2 + \ldots)$$

$$= \frac{1}{2}(2 + 6z + 12z^2 + \ldots)$$

$$= 1 + 3z + 6z^2 + \ldots$$

6. $f(z) = \dfrac{1}{3-z}$, $z_0 = 2$

Let $w = z-2 \Rightarrow z = w+2$ Therefore,

$$\frac{1}{3-z} = \frac{1}{3-w-2} = \frac{1}{1-w} = 1 + w + w^2 + \ldots = 1 + (z-2) + (z-2)^2 + \ldots$$