Lecture 22: Laurent Series

Example:

$$f(z) = \frac{1}{(z-1)(z-2)}$$

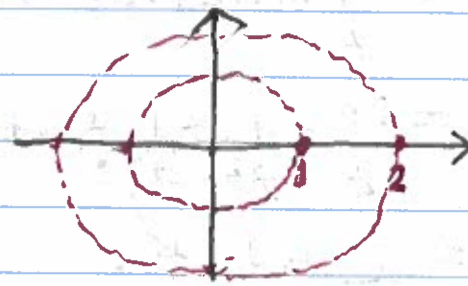
We want series expansions about $z=0$ for different values of z.

We can use partial fractions to simplify:

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}, \quad A=-1,\ B=1.$$

$$\Rightarrow f(z) = -\frac{1}{z-1} + \frac{1}{z-2}$$

$$= \frac{1}{1-z} - \frac{1}{2(1-z/2)}$$



Case 1: $(0 \le |z| < 1)$

$$f(z) = 1 + z + z^2 + \dots - \tfrac{1}{2}\left(1 + z/2 + z^2/4 + \dots\right)$$

$$= \tfrac{1}{2} + \tfrac{3}{4}z + \tfrac{7}{8}z^2 + \dots$$

Case 2: $(1 \le |z| \le 2)$

$$f(z) = \frac{1}{1-z} - \frac{1}{2(1-z/2)}$$

$$= \frac{1}{z(1/z - 1)} - \frac{1}{2(1-z/2)}$$

$$= -\frac{1}{z(1-1/z)} - \frac{1}{2(1-z/2)}$$

$$= -\frac{1}{z}\left(1 + 1/z + 1/z^2 + \dots\right) - \frac{1}{2}\left(1 + z/2 + z^2/4 + \dots\right)$$

$$= \underbrace{\dots - \frac{1}{z^3} - \frac{1}{z^2} - \frac{1}{z} - \frac{1}{2} - \frac{z}{2} - \frac{z^2}{8} + \dots}_{\text{Laurent Series.}}$$

Case 3: $(|z| \geq 2)$

$$f(z) = \frac{1}{1-z} - \frac{1}{2}\frac{1}{1-z/2}$$

$$= \frac{-1}{z(1-1/z)} - \frac{1}{2z}\left(\frac{1}{1/z - 1/2}\right)$$

$$= \frac{-1}{z(1-1/z)} + \frac{1}{z}\left(\frac{1}{1-2/z}\right)$$

$$= \frac{-1}{z}(1 + 1/z + 1/z^2 + \dots) + \frac{1}{z}(1 + 2/z + 4/z^2 + \dots)$$

$$= \frac{-1}{z} - \frac{1}{z^2} - \frac{1}{z^3} + \dots + \frac{1}{z} + \frac{2}{z^2} + \frac{4}{z^3} + \dots$$

$$= \frac{1}{z^2} + \frac{3}{z^3} + \dots$$

Theorem - Let f be analytic in the annulus $r < |z-z_0| < R$. Then f can be expressed as the sum of two series

$$f(z) = \sum_{j=0}^{\infty} a_j (z-z_0)^j + \sum_{j=1}^{\infty} a_{-j} (z-z_0)^{-j}$$

One way to find a_j:

$$\int_\Gamma \frac{f(z)\,dz}{(z-z_0)^{k+1}} = \sum_{j=0}^{\infty} \int_\Gamma a_j (z-z_0)^{j-k-1} + \sum_{j=1}^{\infty} a_{-j} (z-z_0)^{-j-k-1}$$

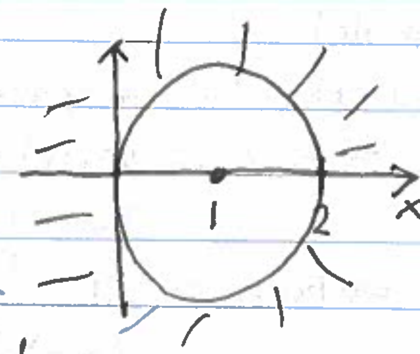
$$\Rightarrow 2\pi i\, a_k = \int_\Gamma \frac{f(z)}{(z-z_0)^{k+1}}\,dz$$

$$\Rightarrow a_k = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{(z-z_0)^{k+1}}\,dz$$

Example:

Find the Laurent series for

$$f(z) = \frac{z^2 - 2z + 3}{z - 2}$$



in the region $|z-1| > 1$. Lets recenter at $z=1$, i.e. set $w = z-1$ and expand.

$$f(w) = \frac{(w+1)^2 - 2(w+1) + 3}{w+1-2} = \frac{w^2 + \cancel{2w} + 1 - \cancel{2w} - 2 + 3}{w-1}$$

$$\Rightarrow f(w) = \frac{w^2+2}{w(1 - 1/w)} = \frac{w^2+2}{w}\left(1 + 1/w + 1/w^2 + 1/w^3 + \ldots\right)$$

$$\Rightarrow f(w) = \left(w + \frac{2}{w}\right)\left(1 + 1/w + 1/w^2 + 1/w^3 + \ldots\right)$$

$$= w + 1 + 1/w + 1/w^2 + 1/w^3 + \ldots$$
$$+ 2/w + 2/w^2 + 2/w^3 + 2/w^4 + \ldots$$

$$= \ldots + \frac{3}{w^4} + \frac{3}{w^3} + \frac{3}{w^2} + \frac{2}{w} + 1 + w + \ldots$$

$$= (z-1) + 1 + \sum_{j=1}^{\infty} \frac{3}{(z-1)^j}$$

Example:

Find the Laurent series for $\tan(z)$ about $z = \pi/2$. Let $w = z - \pi/2$.

$$\Rightarrow \tan(z) = \frac{\sin(z)}{\cos(z)} = \frac{\sin(w+\pi/2)}{\cos(w+\pi/2)} = -\frac{\cos(w)}{\sin(w)}.$$

$$\Rightarrow \tan(z) = -\frac{(1 - w^2/2! + w^4/4! + \ldots)}{(w - w^3/3! + w^5/5! + \ldots)}$$

$$= -\frac{(1 - w^2/2! + w^4/4! + \ldots)}{w(1 - w^3/3! + w^4/5! + \ldots)}$$

$$= -\frac{(1 - w^2/2! + w^4/4! + \ldots)}{w}\left[1 + (w^2/3! - w^4/5! + \ldots) + (w^2/3! - w^4/5! + \ldots)^2 + \ldots\right]$$

$$= -\frac{1}{w}\left(1 + \left(\frac{1}{3!} - \frac{1}{2!}\right)w^2 + \ldots\right)$$

$$= \frac{-1}{w} + \frac{2}{3}w + \ldots$$

$$= \frac{-1}{z - \pi/2} + \frac{2}{3}(z - \pi/2) + \ldots$$