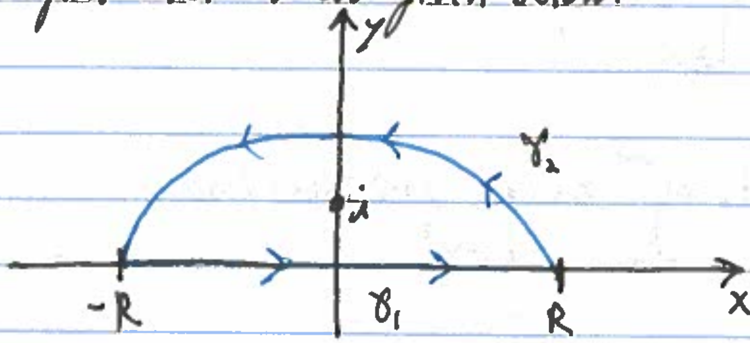
Lecture 26: Improper Integrals with Trig

Example:

$$\int_{-\infty}^{\infty} \frac{x\sin(x)}{1+x^2}\,dx$$

Let $f(z) = \dfrac{ze^{iz}}{1+z^2} = \dfrac{z(\cos(z)+i\sin(z))}{1+z^2}$. Consider the contour integral over γ as given below.



Therefore,

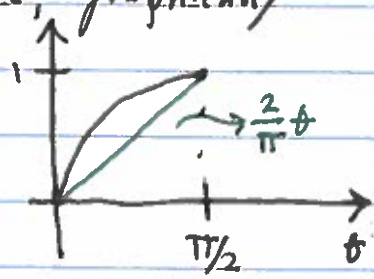
$$\left|\int_{\gamma_2} f(z)\,dz\right| \le \int_0^{\pi} \frac{|Re^{i\theta}e^{iRe^{i\theta}}Rie^{i\theta}|}{|1+R^2e^{2i\theta}|}\,d\theta$$

$$\le \int_0^{\pi} \frac{R^2e^{-R\sin\theta}}{R^2-1}\,d\theta$$

We need to estimate $\int_0^{\pi} e^{-R\sin\theta}\,d\theta$. By symmetry

$$\int_0^{\pi} e^{-R\sin\theta}\,d\theta = 2\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta$$

Note, graphically:



$\Rightarrow \dfrac{2}{\pi}\theta \le \sin\theta$ on $[0,\pi/2]$ by convexity.

$\Rightarrow e^{-R\sin\theta} \le e^{-2R/\pi\,\theta}$

This is known as Jordan's Lemma: $\int_0^{\pi} e^{-R\sin\theta}\,d\theta \le 2\int_0^{\pi/2} e^{-2/\pi R\theta} = \dfrac{C}{R}$

Therefore,

$$\left|\int_{\gamma_2} f(z)\,dz\right| \leq \int_0^{\pi} \frac{R^3}{CR(R^3-1)}\,d\theta = \frac{\pi R^2}{CR(R^2-1)}$$

and thus

$$\lim_{R\to\infty} \int_{\gamma_2} f(z)\,dz = 0.$$

Moreover,

$$\operatorname{Res}(f; i) = \left.\frac{ze^{iz}}{i+z}\right|_{z=i} = \frac{1}{2}e^{-1}$$

Putting it all together:

$$\frac{2\pi i e^{-1}}{2} = \int_{-\infty}^{\infty} \frac{xe^{ix}}{1+x^2}\,dx = \int_{-\infty}^{\infty} \frac{x(\cos(x)+i\sin(x))}{1+x^2}\,dx$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x\sin(x)}{1+x^2}\,dx = \pi e^{-1}, \quad \int_{-\infty}^{\infty} \frac{x\cos(x)}{1+x^2}\,dx = 0.$$